Philippine Journal of Science
149 (2): 269-277, June 2020
ISSN 0031 - 7683
Date Received: 15 Oct 2019

Some Constructions of 3-minimal Graphs with Cycles

Wielson M. Factolerin[1]* and Jean O. Loyola[1]

[1]Institute of Mathematical Sciences and Physics (IMSP),
College of Arts and Sciences University of the Philippines Los Banos (UPLB),
College, Laguna 4031 Philippines

Prime graphs with triangles – namely (a) $P_k \cup \{i(i+2)\}$ with $k \geq 5$ and $2 \leq i \leq k-3$, (b) Q_k with $k \geq 5$, and (c) $S_{k,m,n}$ with an additional edge to form a triangle – were constructed and shown 3-minimal for some vertex-subsets. If G has a triangle and is 3-minimal for a nonstable subset X of $V(G)$, it was shown that G is isomorphic to either $P_5 \cup \{24\} \cong Q_5$ or $P_k \cup \{(k-3)(k-1)\}$. If G has a triangle and is 3-minimal for a stable subset X of $V(G)$, with $A \subseteq V(G)$ such that $G[A]$ is $P_5 \cup \{24\} \cong Q_5$, then either $X \cap A = \emptyset$ or $X \cap A \neq \emptyset$. If $X \cap A = \emptyset$, it was shown that G is isomorphic to one of the forms of $S_{k,m,n}$ with an additional edge to form a triangle. If $X \cap A \neq \emptyset$, it was shown that G is isomorphic to one of the following: (a) one of the forms of $S_{k,m,n}$ with an additional edge to form a triangle; (b) $P_k \cup \{i(i+2)\}$ with $k \geq 6$ and $2 \leq i \leq k-3$; and (c) Q_k with $k \geq 5$.

Keywords: graphs, indecomposable, k-minimal, minimal, prime

INTRODUCTION

In a graph $G = (V(G), E(G))$, a subset M of $V(G)$ is a *module* if every vertex outside M is either adjacent to all or no vertices of M. A graph G is *indecomposable* if all of its modules are trivial: $\emptyset$, $\{v\}$ for all $v \in V(G)$, and $V(G)$. An indecomposable graph of at least three vertices is said to be *prime*.

A *subgraph* F of G is the graph $F = (V(F), E(F))$ where $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. For $X \subseteq V(G)$, the subgraph of G induced by X, $G[X]$, is the graph with $V(G[X]) = X$ such that two vertices are adjacent in $G[X]$ if and only if they are adjacent to G. The subgraph of G induced by $V(G) - X$ is denoted by $G - X$. For $v \in V(G)$, $G - v$ denotes the subgraph of G induced by $V(G) - \{v\}$. The *neighborhood* of a vertex $u \in G$, denoted by $N_G(u)$, is the set of all vertices adjacent to u in G. The *degree* of a vertex $u \in G$, denoted by $\deg_G(u)$, is the number of elements of $N_G(u)$.

A prime graph G is *minimal* for a subset U of $V(G)$ if no proper induced subgraph of G containing U is prime. A prime graph G is k*-minimal* for a subset X of $V(G)$ if it is minimal for X and X has k vertices. A subset X of $V(G)$ is *stable* if its elements are pairwise nonadjacent; otherwise, X is nonstable. Cournier and Ille (1997, 1998) studied indecomposable graphs, directed or undirected, which are 1- or 2-minimal. Alzohairi and Boudabbous (2014, 2015) pursued 3-minimal triangle-free graphs and 4-minimal triangle-free graphs for some nonstable sets and they counted the number of such nonisomorphic graphs given the number of vertices using integer partitions. The stable case remains open. This paper deals with 3-minimal graphs with triangles.

*Corresponding Author: wmfactolerin@up.edu.ph
AMS Subject Classifications: 97K30

For brevity, given a graph G and distinct vertices u and v such that uv is not an edge in G, the graph G with the edge uv is denoted by $G \cup \{uv\}$. Moreover, 12 is the edge joining the vertices 1 and 2. Similarly, $i(i+1)$ is the edge joining the vertices i and $i+1$.

Preliminaries

This section contains definitions and theorems essential to the study. Only uncommon graphs are defined here.

Definition 1 [Ehrenfeucht and Rozenberg (1990), as cited by Alzohairi and Boudabbous (2014)].

a. The graph $S_{k,m,n}$ is the graph with $k+m+n+1$ vertices that is the union of three paths P_k, P_m, and P_n having a common endpoint r.
b. The graph Q_k is the graph P_k with additional edges joining the vertex 2 to $4, 5, \ldots$, and $k-1$.

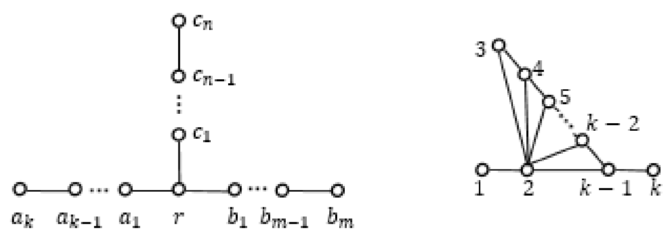

Figure 1. $\boldsymbol{S_{k,m,n}}$ (left) and $\boldsymbol{Q_k}$ (right).

Theorem 1 (Cournier and Ille 1998; Alzohairi and Boudabbous 2014).

a. P_k is prime for all $k \geq 5$.
b. Q_k is prime for all $k \geq 5$.
c. $S_{k,m,n}$ is prime for all $k, m \geq 2$, for all n.

Definition 2 [Ehrenfeucht and Rozenberg (1990), as cited by Alzohairi and Boudabbous (2014)]. For a graph G and $X \subseteq V(G)$ such that $G[X]$ is prime, define the following subsets of $V(G) - X$.

a. $Ext(X)$ is the set of all y outside X such that $G[X \cup \{y\}]$ is prime.
b. $\langle X \rangle$ is the set of all y outside X such that X is a module of $G[X \cup \{y\}]$.
c. For each $u \in X$, $X(u)$ is the set of all y outside X such that $\{u, y\}$ is a module of $G[X \cup \{y\}]$.

Theorem 2 [Ehrenfeucht and Rozenberg (1990), as cited by Alzohairi and Boudabbous (2014)]. Let X be a proper vertex subset of a prime graph G such that $G[X]$ is prime.

a. The family of nonempty sets among $Ext(X)$, $\langle X \rangle$ and $\{X(u) | u \in X\}$ forms a partition of $V(G) - X$.
b. For distinct elements $y, z \in Ext(X)$, the subgraph $G[X \cup \{y, z\}]$ is decomposable if and only if $\{y, z\}$ is a module of $G[X \cup \{y, z\}]$.
c. Given $u \in X$, for $y \in X(u)$ and for z outside $X \cup X(u)$, the subgraph $G[X \cup \{y, z\}]$ is decomposable if and only if $\{y, u\}$ is a module of $G[X \cup \{y, z\}]$.
d. For $y \in \langle X \rangle$ and for z outside $X \cup \langle X \rangle$, the subgraph $G[X \cup \{y, z\}]$ is decomposable if and only if $X \cup \{z\}$ is a module of $G[X \cup \{y, z\}]$.

RESULTS

In this section, prime graphs with triangles are constructed and then shown to be 3-minimal.

Theorem 3. Let $k \geq 5$ and $2 \leq i \leq k-3$. Then, $P_k \cup \{i(i+2)\}$ is prime.

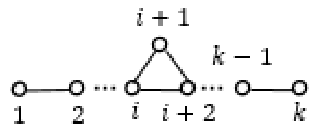

Figure 2. $\boldsymbol{P_k \cup \{i(i+2)\}}$ **with** $\boldsymbol{k \geq 5}$ **and** $\boldsymbol{2 \leq i \leq k-3}$**.**

Proof. By Theorem 1, $P_5 \cup \{24\} \cong Q_5$ is prime.

Let G be the graph $P_k \cup \{i(i+2)\}$, where $k \geq 6$ and $2 \leq i \leq k-3$, and $X = V(G) - \{i+1\}$. By Theorem 1, $G[X]$ is prime since it is isomorphic to P_{k-1}. Now, by Definition 2, $i+1 \notin \langle X \rangle$ since X is not a module of $G = G[X \cup \{i+1\}]$. For each $u \in X$, $\{u, i+1\}$ is not a module of G so $i+1 \notin X(u)$ by Definition 2. By Theorem 2a, $i+1 \in Ext(X)$ and that G is prime.□

Theorem 4. The graph $P_k \cup \{i(i+2), j(j+2)\}$ where $2 \leq i \leq k-6$, $i+3 \leq j \leq k-3$ and $\{i, i+1, i+2\} \neq \{j, j+1, j+2\}$ is prime.

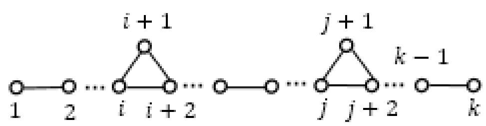

Figure 3. $\boldsymbol{P_k \cup \{i(i+2), j(j+2)\}}$**.**

Proof. Let G be the graph $P_k \cup \{i(i+2), j(j+2)\}$ and $X = V(G) - \{j+1\}$. By Theorem 2, $G[X]$ is prime since it is isomorphic to $P_{k-1} \cup \{i(i+2)\}$. Now, by Definition 2, $j+1 \notin \langle X \rangle$ since X is not a module of $G = G[X \cup \{j+1\}]$. For each $u \in X$, $\{u, j+1\}$ is not a module of G by Definition 2 so $j+1 \notin X(u)$. By Theorem 2a, $j+1 \in Ext(X)$ and that G is prime.□

Remark. For any positive integer n, the graph $S_{1,1,n} \cup \{a_1 b_1\}$ is not prime since $\{a_1, b_1\}$ is a module.

Lemma 1. $S_{2,2,1} \cup \{a_1 b_1\}$ is prime.

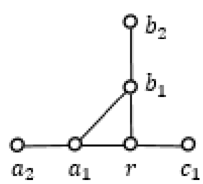

Figure 4. $\boldsymbol{S_{2,2,1} \cup \{a_1 b_1\}}$**.**

Proof. Let G be the graph $S_{2,2,1} \cup \{a_1 b_1\}$ and $X = V(G) - \{c_1\}$. By Theorem 3, $G[X]$ is prime since it is isomorphic to $P_5 \cup \{24\}$. Now, by Definition 2, $c_1 \notin \langle X \rangle$ since X is not a module of $G = G[X \cup \{j+1\}]$. For each $u \in X$, $\{u, c_1\}$ is not a module of G by Definition 2 so $c_1 \notin X(u)$. By Theorem 2a, $c_1 \in Ext(X)$ and that G is prime.□

Lemma 2. $S_{2,2,n} \cup \{a_1 b_1\}$ is prime.

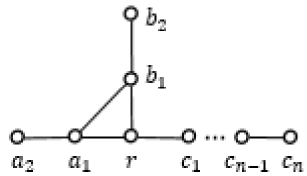

Figure 5. $S_{2,2,n} \cup \{a_1b_1\}$.

Proof. By Lemma 1, Lemma 2 holds when $n = 1$. Now, let n be an integer greater than or equal to 2.

Let G be $S_{2,2,n} \cup \{a_1b_1\}$. Suppose $S_{2,2,n-1} \cup \{a_1b_1\}$ is prime. Let the vertex set of this graph be X. By Definition 2, $c_n \notin \langle X \rangle$ and for any $u \in X$, $\{u, c_n\}$ is not a module of $G = G[X \cup \{c_n\}]$, that is, $c_n \notin X(u)$. Thus, by Theorem 2a, $c_n \in Ext(X)$ and that G is prime for all natural numbers n.□

Lemma 3. Let $k \geq 2$. Then, $S_{k,2,n} \cup \{a_1b_1\}$ is prime.

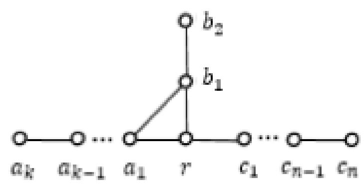

Figure 6. $S_{k,2,n} \cup \{a_1b_1\}$.

Proof. By Lemma 2, Lemma 3 holds when $k = 2$. Now, let k be an integer greater than or equal to 3.

Let G be $S_{k,2,n} \cup \{a_1b_1\}$. Suppose $S_{k-1,2,n} \cup \{a_1b_1\}$ is prime. Let the vertex set of this graph be X. By Definition 2, $a_k \notin \langle X \rangle$ and for any $u \in X$, $\{u, a_k\}$ is not a module of $G = G[X \cup \{a_k\}]$, that is, $a_k \notin X(u)$. Thus, by Theorem 2a, $a_k \in Ext(X)$ and that G is prime for all natural numbers $k \geq 2$.□

Theorem 5. Let $k, m \geq 2$. Then, $S_{k,m,n} \cup \{a_1b_1\}$ is prime.

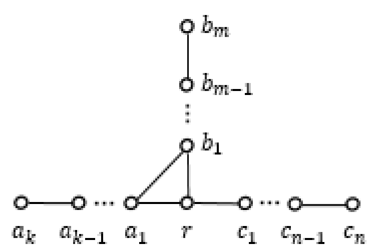

Figure 7. $S_{k,m,n} \cup \{a_1b_1\}$ with $k, m \geq 2$.

Proof. By Lemma 3, the theorem holds when $m = 2$. Now, let m be an integer greater than or equal to 3.

Let G be $S_{k,m,n} \cup \{a_1b_1\}$. Suppose $S_{k,m-1,n} \cup \{a_1b_1\}$ is prime. Let the vertex set of this graph be X. Now, by Definition 2, $b_m \notin \langle X \rangle$ and for any $u \in X$, $\{u, b_m\}$ is not a module of $G = G[X \cup \{b_m\}]$, that is, $b_m \notin X(u)$. Thus, by Theorem 2a, $b_m \in Ext(X)$ and that G is prime for all natural numbers $m \geq 2$.□

Theorem 6. Let $k, j \geq 2$, $m \geq 5$, and $m - 3 \geq j$. Then, $S_{k,m,n} \cup \{b_jb_{j+2}\}$ is prime.

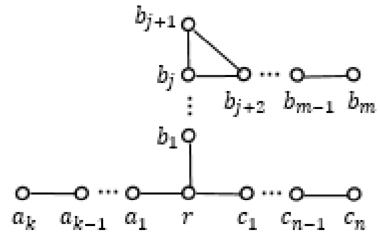

Figure 8. $S_{k,m,n} \cup \{b_j b_{j+2}\}$ with $k, j \geq 2$, $m \geq 5$, and $m - 3 \geq j$.

Proof. Let G be $S_{k,m,n} \cup \{b_j b_{j+2}\}$, with $k \geq 2$ and $m \geq 5$. Let $X = V(G) - \{b_{j+1}\}$. Then, $G[X]$ is prime being isomorphic to $S_{k,m-1,n}$, which is prime by Theorem 1. Now, by Definition 2, $b_{j+1} \notin \langle X \rangle$, and for any $u \in X$, $\{u, b_{j+1}\}$ is not a module of $G = G[X \cup \{b_{j+1}\}]$, then $b_{j+1} \notin X(u)$. Therefore, $b_{j+1} \in Ext(X)$ and, by Theorem 2a, G is prime.□

In Figure 3, we can see that the prime graph $P_k \cup \{i(i+2), j(j+2)\}$ can be constructed from P_k by adding the edges $i(i+2)$ and $j(j+2)$. Similarly, the graphs in Figures 2 and 5–8 are also prime even if edges are added to form isolated triangles in such a way that P_k with the additional edges is a subgraph of the graph.

The succeeding theorems provide sufficient conditions for 3-minimal graphs with triangles.

Theorem 7. $P_5 \cup \{24\}$ is minimal for $\{x, y, z\} \in \{\{1,2,3\}, \{3,4,5\}, \{2,3,4\}, \{1,3,5\}, \{2,3,5\}, \{1,3,4\}\}$.

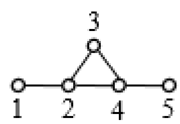

Figure 9. $P_5 \cup \{24\}$.

Proof. This follows directly by definition of a 3-minimal graph for each of the given sets.□

Theorem 8. Let $k \geq 6$ and $2 \leq i \leq k-3$. Then, $P_k \cup \{i(i+2)\}$ is minimal for $\{1, i+1, k\}$.

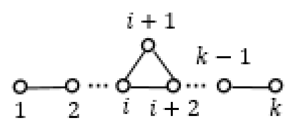

Figure 10. $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$.

Proof. Any proper subgraph induced by a vertex-subset containing $\{1, i+1, k\}$ is disconnected and has at least three vertices. From this disconnected graph, take a connected subgraph as a nontrivial module. Then, the theorem follows by definition of a minimal graph.□

From the proof of Theorem 8, it can be deduced that a disconnected graph with at least three vertices is decomposable.

Theorem 9. $P_k \cup \{(k-3)(k-1)\}$ is minimal for $\{1, k-2, k-1\}$.

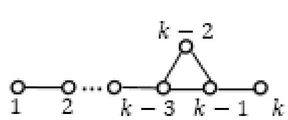

Figure 11. $P_k \cup \{(k-3)(k-1)\}$.

Proof. Consider any proper subgraph induced by a vertex-subset containing $\{1, k-2, k-1\}$. If k is the only vertex not included in the subgraph, then the subgraph is decomposable having $\{k-2, k-1\}$ as a nontrivial module.

In any other case, the subgraph is disconnected and has at least three vertices. Hence, the subgraph is decomposable.

The theorem follows by definition of a minimal graph.□

The proof of Theorem 10 follows an argument similar to the proof of Theorems 8 and 9 and is omitted.

Theorem 10.

a. Let $k, m \geq 2$. $S_{k,m,n} \cup \{a_1 b_1\}$ is minimal for $\{a_k, b_m, c_n\}$.
b. Let $k, j \geq 2$, $m \geq 5$, and $m - 3 \geq j$. $S_{k,m,n} \cup \{b_j b_{j+2}\}$ is minimal for $\{a_k, b_m, c_n\}$.

Theorem 11. Let $k \geq 5$. Then, Q_k is minimal for $\{1,3,k\}$. For each nonempty subset B of $V(Q_k) - \{1,3,k\}$, $Q_k - B$ is prime.

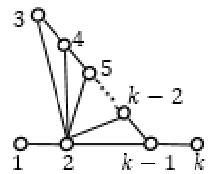

Figure 12. Q_k.

Proof. By Theorem 1, Q_k is prime. By Theorem 7, Q_5 is minimal for $\{1,3,5\}$. Let $k > 5$.

Let B be a nonempty subset of $V(Q_k) - \{1,3,k\}$. $Q_k - B$ is an arbitrary proper induced subgraph of Q_k containing $\{1,3,k\}$.

If both 2 and $k - 1$ are not in B, then both are in $Q_k - B$. Here, $Q_k - B$ has a nontrivial module $\{1,3,4, \ldots, k-3\} - \{j | j \in B\}$.

Otherwise, $Q_k - B$ is disconnected with at least three vertices. Take a connected subgraph to be a nontrivial module of $Q_k - B$.

In either case, the proper induced subgraph $Q_k - B$, which contains $\{1,3,k\}$, is decomposable.

The theorem follows by definition of a minimal graph.□

We now present some applications of the 3-minimal graphs shown in Theorems 7–11.

Theorem 12. Let X be a 3-vertex subset in a graph G with a triangle such that X is nonstable subset of $V(G)$. If G is minimal for X, then G is isomorphic to either $P_5 \cup \{24\}$ or $P_k \cup \{(k-3)(k-1)\}$.

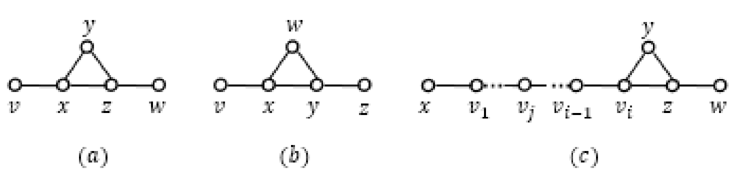

Figure 13. Illustration for the Proof of Theorem 12.

Proof. Let $A = \{x, y, z\}$ be a nonstable subset of G such that G is minimal for A. Since A is nonstable, $|E(G[A])| = 3, 2,$ or 1. Since G is minimal for A, then G must be prime.

If $G[A]$ has three edges, then there exist $v, w \in V(G)$ such that $N(v) = \{x\}$ and $N(w) = \{z\}$ so that $G = G[A \cup \{v, w\}]$ is prime. Hence, $G \cong P_5 \cup \{24\}$. (Figure 13a)

Suppose $G[A]$ has exactly two edges. Without loss of generality, assume that $xz \notin E(G)$. Then, there exist $v, w \in V(G)$ such that $N(v) = \{x\}$ and $N(w) = \{x, y\}$ so that $G = G[A \cup \{v, w\}]$ is prime. Hence, $G \cong P_5 \cup \{24\}$. (Figure 13b)

Suppose $G[A]$ has exactly one edge. Without loss of generality, assume that $xy, xz \notin E(G)$ and $yz \in E(G)$. If $x, v_1, \dots, v_{i-1}, v_i, y$, and z are all the vertices of G, and since G has a triangle, there exists a sequence of vertices $v_1, \dots, v_i$, $i \geq 1$ such that $G[(x, v_1, \dots, v_i, y)]$ and $G[(x, v_1, \dots, v_i, z)]$ are paths. Then G is not prime since it has a nontrivial module $\{y, z\}$. Since G is prime, $V(G)$ has at least one more vertex w.

If w is adjacent to one of $x, v_1, \dots, v_{i-1}$, or v_i, then G has a nontrivial module and is not prime. If w is adjacent to either y or z, then we are done. Without loss of generality, assume that w is adjacent to z. If there exists another $w' \in V(G)$ adjacent to any one of $x, v_1, \dots, v_{i-1}, v_i, y, z$, and w, then G is not minimal for A. Thus, $V(G)$ has exactly one more vertex w such that $N_G(w) = \{z\}$. Hence $G = G[A \cup \{v_1, \dots, v_i, w\}] \cong P_k \cup \{(k-3)(k-1)\}$. (Figure 13c) □

Theorem 13. Let X be a 3-vertex subset in a graph G with a triangle such that X is a stable subset of G and G is minimal for X. Let $A \subseteq V(G)$ such that $G[A]$ is $P_5 \cup \{24\} \cong Q_5$. If $X \cap A = \emptyset$, then G is isomorphic to one of the forms of $S_{k,m,n}$ with an additional edge given in Theorem 10.

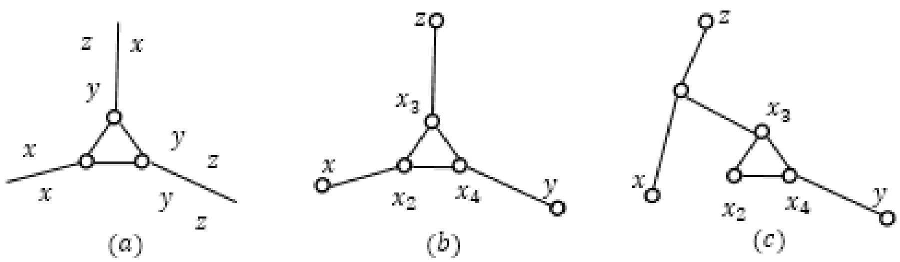

Figure 14. Illustration for the Proof of Theorem 13.

Proof. Let $X = \{x, y, z\}$, $A = \{x_1, x_2, x_3, x_4, x_5\}$. Since G is minimal for X, then G must be prime. G must also be connected or else G is decomposable. Since $X \cap A = \emptyset$, then $|X \cap A| = 0$.

If G has a path containing x, y, and z (Figure 14a), the subgraph containing this path and the remaining vertex that will form the triangle in A is isomorphic to $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$, which is prime, and is either a prime proper subgraph of G or G itself. If this subgraph is G itself then G is not minimal for X, a contradiction. If this subgraph containing X is a prime proper subgraph of G, then G is not minimal for X by definition, a contradiction. Thus, a path in G can only contain two of x, y, and z, say x and y. An xy–path containing two vertices of $\{x_2, x_3, x_4\}$, say x_2 and x_4, exists since G is connected. Similarly, there exists a yz–path containing x_4 and x_3. If there is an xz–path containing x_2 and x_3, then the graph containing the xy–, yz– and xz–paths described is isomorphic to $S_{k,m,n} \cup \{a_1 b_1\}$ (Figure 14b). If there is no xz–path containing x_2 and x_3, then the graph containing the xy–, yz– and xz–paths described is isomorphic to $S_{k,m,n} \cup \{b_j b_{j+2}\}$, $j \leq m-3$ (Figure 14c). In any case, G is isomorphic to one of the forms of $S_{k,m,n}$ with an additional edge. □

Theorem 14. Let X be a 3-vertex subset in a graph G with a triangle such that X is a stable subset of $V(G)$ and G is minimal for X. Let $A \subseteq V(G)$ such that $G[A]$ is $P_5 \cup \{24\} \cong Q_5$. If $X \cap A \neq \emptyset$ then G is isomorphic to one of the following:

a. One of the forms of $S_{k,m,n}$ with an additional edge given in Theorem 10;
b. $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$;
c. Q_k, $k \geq 5$.

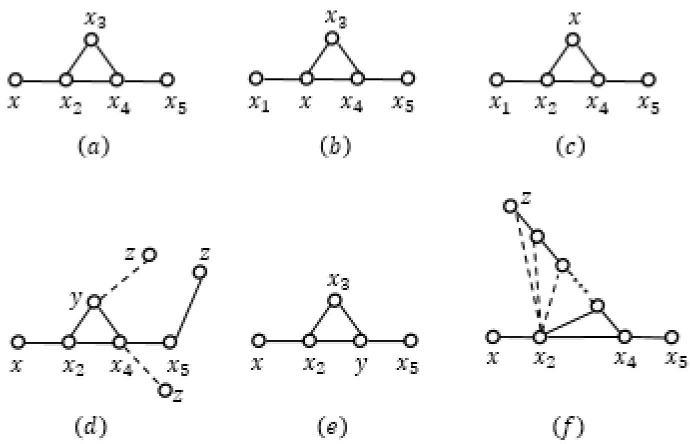

Figure 15. Illustration for the Proof of Theorem 14.

Proof. There are three possibilities for $X \cap A$: $|X \cap A| = 1$, 2, or 3.

For $|X \cap A| = 1$: Let $x \in X \cap A$. There are three non-analogous cases for $x \in A$: $x = x_1$, $x = x_2$, or $x = x_3$.

Case 1 (for $x = x_1$; see Figure 15a). Suppose $deg_G(x) \geq 2$. Then there exists a vertex x' in G such that $x'x \in E(G)$. Similar to the proof of Theorem 13, since G is prime, there cannot exist a path containing x', y, and z and two vertices of the triangle $G[\{x_2, x_3, x_4\}]$ in A. Then, such a path must only contain two of x', y, and z. This graph is minimal for $\{x', y, z\}$ but not for X since $G - x'$ is prime. Thus, $x = x'$ and $deg_G(x) = 1$. An xz-path must not contain y or else, this graph is not minimal for X since $G - x_3$ is a path and is prime. Thus, an xy-path and an xz-path must differ by some number of vertices (and edges). If an xy-path and an xz-path both contain the edge x_2x_4, then G – which consists of A together with the chosen xy- and xz-paths – is isomorphic to one of the forms of $S_{k,m,n}$ with an additional edge. If an xy-path contains the edge x_2x_3 but not the vertex x_4 and an xz-path contains the edge x_2x_4 but not the vertex x_3, then there exists a yz-path containing the edge x_3x_4 by the preceding case. Then, G is isomorphic to $S_{2,m,n} \cup \{a_1b_1\}$.

Case 2 (for $x = x_2$; see Figure 15b). Let $x' = x_1$. Then, similar to the proof of Theorem 13, since G is prime, there cannot exist a path containing x', y, and z and two vertices of the triangle $G[\{x_2, x_3, x_4\}]$ in A. Then, such a path must only contain two of x', y, and z. Also, there exists an $x'z$-path containing the edge xx_4, an $x'y$-path containing the edge xx_3, and a yz-path containing the edge x_3x_4. Then, G consisting of these paths is isomorphic to $S_{2,m,n}$, which is not minimal for X since $G - x_1$ is isomorphic to $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$, which is prime. Thus, this case is not possible.

***Case 3* (for $x = x_3$; see Figure 15c).** G cannot be isomorphic to $S_{2,m,n} \cup \{a_1b_1\}$ such that $x = a_1$ and $x_2 = b_1$ since this graph is not minimal for X. Thus, $N_G(x) = \{x_2, x_4\}$. This graph is isomorphic to $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$ and is only minimal for a stable set X when $y = 1$ and $z = k$.

For $|X \cap A| = 2$: Let $x, y \in X \cap A$. There are three stable and non-analogous cases for $x, y \in A$: $x = x_1$, $y = x_3$; $x = x_1$, $y = x_4$; or $x = x_1$, $y = x_5$.

***Case 1* (for $x = x_1, y = x_3$; see Figure 15d).** Suppose $deg_G(y) \geq 3$. If there exists an xz-path containing the edge x_2y but not the vertex x_4, then G is not minimal for X since $G - \{x_4, x_5\}$ is a path. But G must be minimal for X so the xz-path must contain x_4 or not contain x_2y. If the xz-path does not contain both x_4 and x_2y then G is not minimal for X. If the xz-path contains both x_2y and x_4 as well as x_5, then the resulting graph G is isomorphic to $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$ with $x = k$, $y = k-2$, and $z = 1$. If the xz-path contains both x_2y and x_4 but not x_5, then the resulting graph G is not minimal for X since $G - x_5$ is prime. Thus, $deg_G(y) = 2$ and G is isomorphic to $P_k \cup \{(k-2)(k-1)\}$.

Case 2 (for $x = x_1$, $y = x_4$; see Figure 15e). If there is a yz-path containing x, then the resulting graph G is not minimal for X since $G - x_3$ is a path that is prime. If there is a yz-path containing x_3, then the resulting graph G is not minimal for X, by definition, since $G - x_5$ is isomorphic to $P_k \cup \{i(i+2)\}$, $k \geq 6$, $2 \leq i \leq k-3$, which is prime. If there is a yz-path containing x_5, then the resulting graph G is not minimal for X, by definition, since $G - x_3$ is a path that is prime. Thus, this case is not possible.

Case 3 (for $x = x_1$, $y = x_5$; see Figure 15f). z cannot be on the same path as the vertices x, x_2, x_4, and y since the resulting graph G is not minimal for X since $G - x_3$ is a path that is prime. Thus, a yz-path must contain x_3 but not x. The resulting graph G is minimal for X and is isomorphic to $S_{2,m,1} \cup \{a_1 b_1\}$, $m \geq 2$ such that $a_1 = x_2$ and $b_1 = x_3$. Furthermore, if $deg_G(x_2) \geq 3$, then G is isomorphic to $S_{2,m,1}$, $m \geq 2$ with an additional edge if $x_2 z \in E(G)$ and $G \cong Q_k$, $k \geq 6$ if $x_2 z \in E(G)$.

For $|X \cap A| = 3$: x, y, and z must be in A. The only case where X is a stable set is when $x = x_1$, $y = x_3$, and $z = x_5$. Then, $G = G[A]$ is minimal for X and is isomorphic to $P_5 \cup \{24\} \cong Q_5$. □

ACKNOWLEDGMENTS

The authors would like to thank UPLB-IMSP for the continued support to its faculty.

REFERENCES

ALZOHAIRI M. 2015. Triangle-free graphs which are minimal for some nonstable 4-vertex subset. Arab Journal of Mathematical Sciences 21: 159–169.

ALZOHAIRI M, BOUDABBOUS Y. 2014. 3-minimal triangle-free graphs. Discrete Math 331: 3–8.

COURNIER A, ILLE P. 1998. Minimal indecomposable graphs. Discrete Math 183: 61–80.

EHRENFEUCHT A, ROZENBERG G. 1990. Primitivity is hereditary for 2-structures. Theoretical Computer Science 70(3): 343–358.

ILLE P. 1997. Indecomposable graphs. Discrete Math 173: 71–78.