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Weak Algebra Bundles and Associator Varieties

Clarisson Rizzie Canlubo

Institute of Mathematics, and National Science Research Institute
College of Science, University of the Philippines Diliman
Quezon City 1101 Philippines

Algebra bundles, in the strict sense, appear in many areas of geometry and physics. However, the structure of an algebra is flexible enough to vary non-trivially over a connected base – giving rise to a structure of a weak algebra bundle. We will show that the notion of a weak algebra bundle is more natural than that of a strict algebra bundle by illustrating that the classifying object of algebra bundles and, consequently, of weak algebra bundles is a weak algebra bundle. We will give necessary and sufficient conditions for weak algebra bundles to be locally trivial. The collection of non-trivial associative algebras of a fixed dimension forms a projective variety called associator varieties. We will show that these varieties play the role the Grassmannians play for principal $O(n)$ −bundles.

Key words: algebra bundles, associator varieties, differential connections

INTRODUCTION

Weak algebra bundles are generalizations of (strict) algebra bundles. They are monoid objects in the category of vector bundles. Algebra bundles appear more frequently in the literature. The exterior bundle and the Clifford bundle are examples of (strict) algebra bundles. In Section 3, we look at the varieties of associative algebras of a fixed dimension – the so-called associator varieties. In Section 4, we will show that weak algebra bundles are more natural than algebra bundles by constructing the so-called classifying weak algebra bundle. In Section 5, we will give necessary and sufficient conditions for a weak algebra bundle to be locally trivial and, hence, strictness. We will introduce the notion of a differential connection. Existence of a differential connection together with a technical condition guarantee local triviality.

A lot has been written for (associative) algebra bundles (Chidambara and Kiranagi 1994, Kiranagi and Rajendra 2008). An almost equal amount of literature has been devoted to Lie algebra bundles (Douady and Lazard 1966). In this article, we restrict to finite-rank weak algebra bundles. Most of the algebra bundles appearing in the literature are of finite rank. These include the aforementioned bundles like the exterior and the Clifford bundles (Husemöller 1994). For the infinite dimensional case, many works have been done (Dadarlat 2006, Fell 1961). The content of this work is a part of my Ph.D. Thesis (Canlubo 2017) under the supervision of Ryszard Nest.

For the purpose of the discussion to follow, let us recall some few basic objects and facts used here. By a *vector bundle E over X of rank n*, we mean a topological space E together with a continuous surjection $E \xrightarrow{p} X$ satisfying:

*Corresponding author: crpcanlubo@math.upd.edu.ph

a) for every $x \in X$, $p^{-1}(x)$ is an n-dimensional vector space over $\mathbb{R}$,
b) there is an open covering $\{U_\alpha | \alpha \in I\}$ of X and homeomorphisms

$p^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times \mathbb{R}^n$ for each $\alpha \in I$ making the following diagram commutes,

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow p & & \downarrow pr_1 \\ U_\alpha & = & U_\alpha \end{array}$$

where pr_1 is the projection onto the first factor, and
c) the restriction of ϕ_α on $p^{-1}(x)$ is a linear isomorphism onto $\{x\} \times \mathbb{R}^n$.

The homeomorphisms ϕ_α are called the *trivializing maps* of the vector bundle $E \xrightarrow{p} X$ relative to the open covering $\{U_\alpha | \alpha \in I\}$. A vector bundle over X that is globally homeomorphic to a space of the form $X \times \mathbb{R}^n$ for some $n \in \mathbb{N}$ is said to be *trivial*.

Pullbacks are used quite frequently in this article. Let us briefly recall what a pullback is. Given objects A, B, C, D of a category $\mathcal{C}$ and arrows $A \longrightarrow B, A \longrightarrow C, C \longrightarrow D$ and $B \longrightarrow D$, we say that A is a *pullback* of B along $C \longrightarrow D$ if the diagram described by the aforementioned arrows commute, and if there is another object A' and arrows $A' \longrightarrow B$ and $A' \longrightarrow C$ that – together with $C \longrightarrow D$ and $B \longrightarrow D$ – give a commutative diagram. Consequently, there is an arrow $A' \longrightarrow A$ that factorizes the arrows $A' \longrightarrow B$ and $A' \longrightarrow C$. We will abuse language by calling A or $A \longrightarrow C$ the pullback of B or $B \longrightarrow D$ along $C \longrightarrow D$.

In section 4, we will show that there is a weak algebra bundle that classifies all rank n weak algebra bundles for $n \in \mathbb{N}$ fixed. This is the analogue of the Grassmannians in relation to principal $O(n)$-bundles. By a *principal $O(n)$-bundle* over X, we mean a continuous surjection $P \xrightarrow{p} X$ such that for any $x \in X$, $p^{-1}(x) \cong O(n)$ as topological groups and P carries an action of $O(n)$ that preserves the fibers $p^{-1}(x), x \in X$. It is known that there is a space $EO(n)$ – defined only up to homotopy – and a continuous surjection (for a chosen homotopy representative) $EO(n) \xrightarrow{q} G(n, \mathbb{R}^\infty)$, which is a principal $O(n)$-bundle satisfying the following universal property: for any principal $O(n)$-bunlde $Y \xrightarrow{p} X$, there is a continuous map $X \xrightarrow{f} G(n, \mathbb{R}^\infty)$ such that Y is the pullback of $EO(n)$ along f (May 1999).

Weak Algebra Bundles

An *algebra bundle* is a vector bundle in which the fibers are algebras rather than just vector spaces and such that the trivialization maps are algebra isomorphisms. It follows immediately that if the base space is connected, then the fiber algebras are mutually isomorphic. For a weak algebra bundle, we do not require that the local trivialization maps are algebra isomorphisms. A *weak algebra bundle* over a space X is a monoid object in the category of real vector bundles $Vec(X)$ over X. More precisely, we have the following definition.

Definition 1. Let X be a topological space. A *weak algebra bundle* $A \xrightarrow{p} X$ is a vector bundle together with bundle map $A \otimes A \xrightarrow{\mu} A$ such that $\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)$. If, in addition, there is a bundle map $\mathbb{I}_X \xrightarrow{\eta} A$ where $\mathbb{I}_X$ is the trivial line bundle over X such that $\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta)$, we say that the weak algebra bundle is *unital*. The map μ is called the *bundle multiplication* or simply the *multiplication*, while the map η is called the *bundle unit* or simply the *unit*.

Note that $\Gamma(X, A\otimes A) \cong \Gamma(X,A)\otimes_{C(X)}\Gamma(X,A)$ and $\Gamma(X,\mathbb{I}_X) \cong C(X)$ are $C(X)$ −bimodules. The global section functor Γ induces a multiplication μ_* and a unit map η_* on $\Gamma(X,A)$, given by:

$$\mu_*(\sigma,\tau)(x) = \sigma(x)\tau(x), \qquad \eta_*(\alpha)(x) = \eta(\alpha(x))$$

for any $\sigma,\tau \in \Gamma(X,A)$, $\alpha \in C(X)$, and $x \in X$. These maps turn $\Gamma(X,A)$ into a unital $C(X)$ −algebra. Conversely, a unital $C(X)$ −algebra structure on $\Gamma(X,A)$ turns $A \xrightarrow{p} X$ into a weak algebra bundle. From this equivalence, we see immediately that (strict) algebra bundles are weak algebra bundles.

Associator Varieties

Despite the name, weak algebra bundles are more natural than (strict) algebra bundles. In this section, we will construct a certain universal weak algebra bundle that describes *all* weak algebra bundles (strict included) of a particular rank. First, let us consider a finite-dimensional vector space A with a chosen basis $\{x_1, x_2, \ldots, x_n\}$. An associative algebra structure on A is completely determined by the structure constants $\alpha_{ij}^k, 1 \le i,j,k \le n$ satisfying:

$$\sum_l \alpha_{ij}^l\, \alpha_{lk}^m - \alpha_{il}^m \alpha_{jk}^l = 0 \tag{1}$$

for all $1 \le i,j,m,k \le n$. Let χ_n be the variety defined by the n^4 equations (1) called the *rank* n *associator variety*. Let ξ_n be the quotient of χ_n by the equivalence relation $x \sim y$ if $A_x \cong A_y$, where A_x means the algebra structure on A corresponding to the set of structure constants$x \in \chi_n$, equipped with the quotient topology. In general, $\xi_n$is not a variety. For $n \in \mathbb{N}$, the *classifying weak algebra bundle* $\hat{\mathcal{A}}$ of rank n is the weak algebra bundle $\hat{\mathcal{A}} \xrightarrow{p} \xi_n$ such that $p^{-1}(x) = A_x$.

Let us look more closely to the associator varieties χ_n. For the purpose of this section, we let χ_n be the variety given by the system of equations (1) with the all-zero solution removed. This makes χ_n a projective variety. Before we go into the analysis of these varieties, let us give some explicit points.

Example 1. For any $1 \le i,j,k \le n$, let

$$\alpha_{ij}^k = \begin{cases} 1 & k = i + j \mod n \\ 0 & otherwise \end{cases}$$

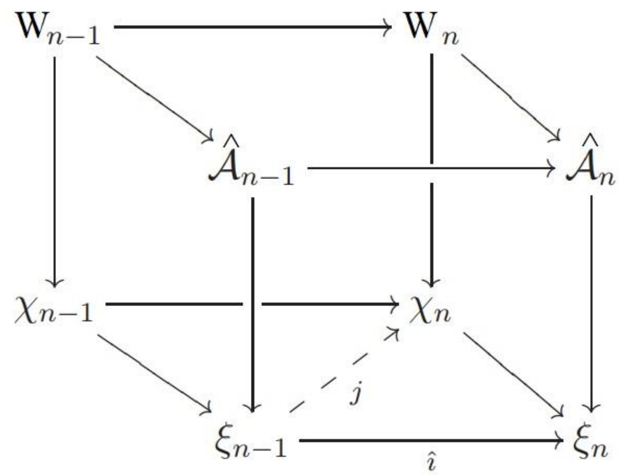
The algebra A_x, where $x = (\alpha_{ij}^k)$, is the truncated polynomial algebra $\mathbb{C}[x]/(x^n)$.

Example 2. For any $1 \le i,j,k \le n$, let $\alpha_{ij}^k = g(k)h(i)h(j)$ where $\{1,2,\ldots,n\} \xrightarrow{g,h} \mathbb{C}$ are arbitrary functions. A particular example is when $g(k) = k$ and $h(j) = e^{j\pi\sqrt{-1}}$.

The associator varieties fit into a natural sequence. The variety χ_{n-1} is the intersection of χ_n with the varieties $\alpha_{in}^k = \alpha_{ni}^k = 0, 1 \le i,k \le n$. This gives an inclusion of varieties $\chi_{n-1} \xrightarrow{i_{n-1}} \chi_n$. This inclusion is natural in the sense of the following proposition.

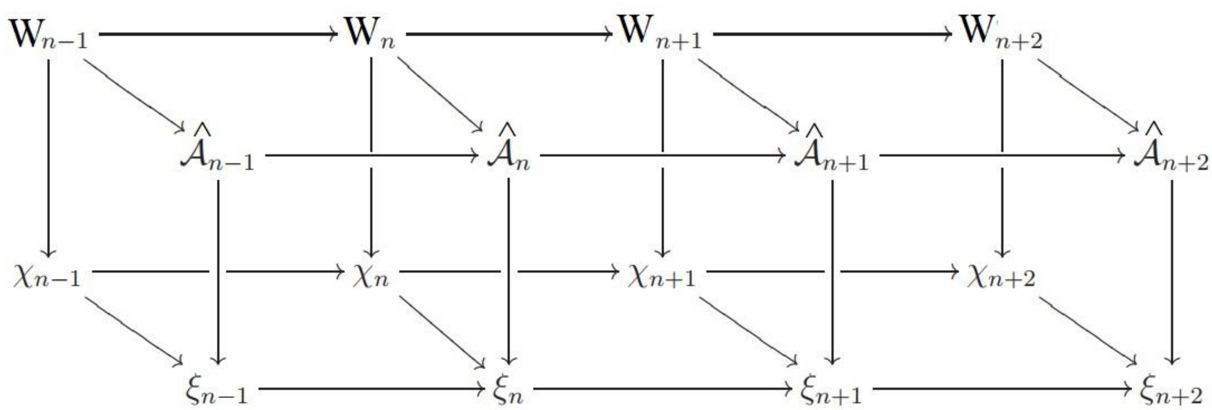
Proposition 1. *The tautological weak algebra bundle* $W_{n-1} \xrightarrow{r_{n-1}} \chi_{n-1}$ *is the pullback of the tautological weak algebra bundle* $W_n \xrightarrow{r_n} \chi_n$ *along the map* i_{n-1}.

Proof. The proposition follows directly from the following pullback cube:



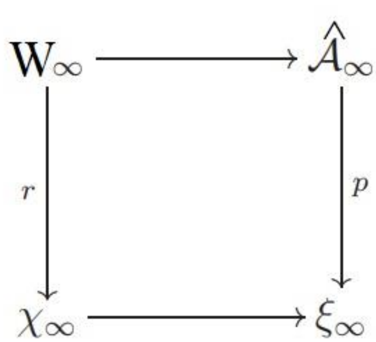
where the map j is the map induced by the universality of the quotient ξ_{n-1} and the map $\hat{\imath}$ is the composition of the j and the quotient map $\chi_n \longrightarrow \xi_n$. ■

By Proposition 1, we have several stratifications given by the horizontal maps in the complex of spaces below:



Using maps above, we can define the spaces $\chi_\infty := \varinjlim \chi_n$. Similarly, we define ξ_∞, W_∞ and $\mathcal{A}_\infty$ as direct limits of the obvious sequence of spaces and maps. Then, it is immediate to check that there are weak algebra bundles $W_\infty \xrightarrow{r} \chi_\infty$ and $\mathcal{A}_\infty \xrightarrow{r} \xi_\infty$. Moreover, these fit into the following pullback square.

where $\chi_\infty \longrightarrow \xi_\infty$ is the direct limit of the quotient maps $\chi_n \longrightarrow \xi_n$. For a concise reference on direct limits, see May (1999)



Let us end this section by looking at the tangent spaces of associator varieties. The following computation can be found in Shafarevich (1994). Consider a point $\alpha \in \chi_n$ given by $\{\alpha_{ij}^k | 1 \leq i,j,k \leq n\}$ in χ_n. Then, tangent vectors $v = \{v_{ij}^k | 1 \leq i,j,k \leq n\}$ to χ_n at the point α satisfy the equation:

$$\sum_l \left(\alpha_{ij}^l v_{lk}^m + \alpha_{lk}^m v_{ij}^l - \alpha_{il}^m v_{jk}^l - \alpha_{jk}^l v_{il}^m\right) = 0$$

Let $\{x_1, \dots, x_n\}$ be a basis for A giving the structure constants $\{\alpha_{ij}^k | 1 \leq i,j,k \leq n\}$. The bilinear function $f_v: A \times A \longrightarrow A$ given by $f_v(x_i, x_j) = \sum_k v_{ij}^k x_k$ satisfies the relation:

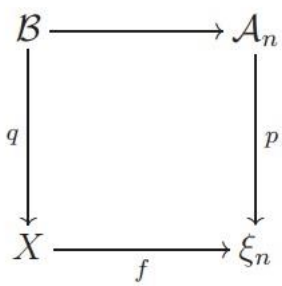
$$x f_v(y,z) - f_v(xy,z) + f_v(x,yz) - f_v(x,y)z = 0 \qquad (2)$$

for all $x, y, z \in A$. Condition (2) is called the cocycle condition. More precisely, bilinear functions f satisfying (2) are called 2-cocycles. The set of all 2-cocycles in A, denoted by $Z^2(A)$ is a vector space. The map $v \mapsto f_v$ defines a linear isomorphism between the tangent space to χ_n at the point α and the space $Z^2(A)$.

Classifying Weak Algebra Bundles

In this section, we will show that the orbivariety χ_n is the analogue the Grassmannians play for principal $O(n)$-bundles. Let us first recall that the Grassmannian variety $G(n, \mathbb{R}^\infty)$ is the *classifying space* for $O(n)$. This in particular means that for any principal $O(n)$-bundle $Y \xrightarrow{p} X$, there is a continuous map $X \xrightarrow{f} G(n, \mathbb{R}^\infty)$ such that Y is the pullback of $EO(n)$ along f. The following theorem justifies the name of the bundle $\mathcal{A}_n \xrightarrow{p} \xi_n$.

Theorem 1. *Let $\mathcal{A}_n \xrightarrow{p} \xi_n$ be the classifying weak algebra bundle of rank n. Let $\mathcal{B} \xrightarrow{q} X$ be a weak algebra bundle of rank n. Then, there is a continuous map $X \xrightarrow{f} \xi_n$ such that $\mathcal{B} \cong f^*\mathcal{A}_n$ as weak algebra bundles i.e., a map f which makes the following diagram a pullback diagram:*



Proof. For a weak algebra bundle $\mathcal{B} \xrightarrow{q} X$, define the map $X \xrightarrow{f} \xi_n$ that sends $x \in X$ to $y \in \xi_n$ if $q^{-1}(x) \cong p^{-1}(y)$. Let $\{U_\alpha | \alpha \in I\}$ be an open cover of X trivializing $\mathcal{B} \xrightarrow{q} X$. Then, $q^{-1}(x) \cong U_\alpha \times B$ where B is the underlying vector space of the typical fiber of q. Let $\{x_1, \dots, x_n\}$ be a basis for B and $\gamma_{ij}^k(x)$ be the structure functions of B_x for $x \in U_\alpha$. Then, the continuous functions $\gamma_{ij}^k: U_\alpha \longrightarrow \mathbb{R}$ satisfy the equations (1). This defines a continuous map $U_\alpha \xrightarrow{f} \xi_n$ sending $x \in U_\alpha$ to $\left(\gamma_{ij}^k(x) | 1 \leq i,j,k \leq n\right) \in \xi_n$. By Proposition 2 below, these functions extend globally to a continuous function $X \xrightarrow{f} \xi_n$. Finally, it is straightforward to check that, indeed, $\mathcal{B} \cong f^*\mathcal{A}_n$. ■

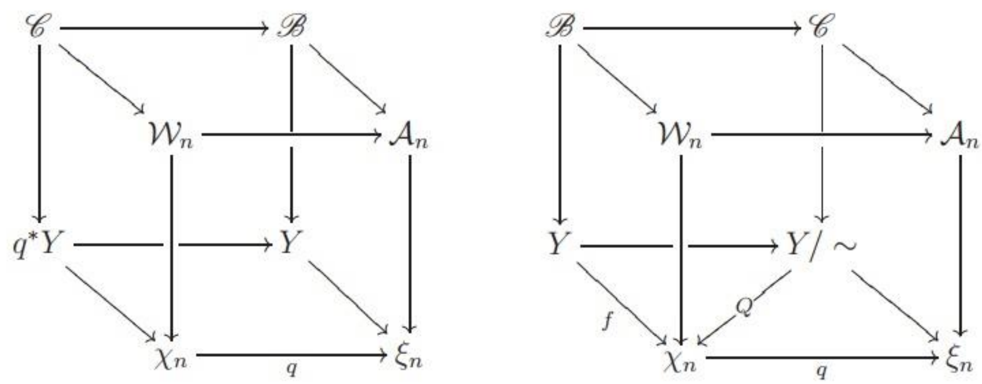
If $\mathcal{B} \xrightarrow{q} X$ is a (strict) algebra bundle, then the map $X \xrightarrow{f} \xi_n$ asserted by Theorem 1 is just the constant map – sending every point of X to the unique point $y \in \xi_n$ such that the typical fiber of $\mathcal{B} \xrightarrow{q} X$ is isomorphic to $p^{-1}(y)$.

In view of Proposition 2 below, the non-triviality of the underlying vector bundle of a rank n weak algebra bundle $\mathcal{B} \xrightarrow{q} X$ is determined by the homotopy type of the function f asserted by Theorem 1 but not completely so. For example, a strict algebra bundle $\mathcal{B} \xrightarrow{q} X$ may have a non-trivial underlying vector bundle and yet the associated function f is constant and hence, homotopically trivial.

Let $W_n \xrightarrow{r} \chi_n$ be the pullback of the rank n classifying weak algebra bundle along the quotient map $\chi_n \xrightarrow{f} \xi_n$. We call this bundle the *tautological weak algebra bundle*. The following theorem illustrates that for most purposes we can use the tautological weak algebra bundle in place of the classifying weak algebra bundle. Denote by $Pull(\mathcal{B}, X)$ the set of all weak algebra bundles that are pullbacks of $\mathcal{B} \xrightarrow{f} X$.

Theorem 2. *There is a canonical bijection between* $Pull(\mathcal{A}_n, \xi_n)$ *and* $Pull(W_n, \chi_n)$.

Proof. If $\mathcal{B} \longrightarrow Y$ is a pullback of the classifying weak algebra bundle, then pulling back maps along appropriate maps as illustrated by the left cube below gives a weak algebra bundle $\mathcal{C} \longrightarrow q^*Y$ where q is the quotient map.



Conversely, let $\mathcal{B} \longrightarrow Y$ be a pullback of the tautological weak algebra bundle along a map $Y \xrightarrow{f} \chi_n$. Let $\sim$ be the equivalence relation on Y defined as $y \sim y'$ if $f(y) = f(y')$. Let Q be the map induced by the universality of the quotient $Y/\sim$ and let $Y/\sim \longrightarrow \xi_n$ be the composition of Q and the quotient map q. Pulling back maps along appropriate maps according to the right cube above gives a weak algebra bundle $\mathcal{C} \longrightarrow Y/\sim$ that is a pullback of the classifying weak algebra bundle. ■

The advantage of working with $W_n \xrightarrow{r} \chi_n$ is the fact that χ_n is a variety $W_n \xrightarrow{r} \chi_n$, which is a regular vector bundle. Let us end this section by a triviality statement regarding the classifying weak algebra bundles.

Proposition 2. *As vector bundles,* $\mathcal{A}_n \xrightarrow{p} \xi_n$ *are parallelizable for all* $n \in \mathbb{N}$.

For $i = 1, \ldots, n$ the sections $\xi_n \xrightarrow{\sigma_i} \mathcal{A}_n, x \mapsto x_i$ give a set of pointwise linearly independent set of n sections. This illustrates parallelizability.

Local Triviality of Weak Algebra Bundles

In this section, we give necessary and sufficient conditions for a weak algebra bundle to be a strict algebra bundle. For this purpose, we will specialize in the smooth case. Let X be a connected smooth manifold.

Definition 2. Let $E \twoheadrightarrow X$ be a smooth vector bundle such that the fibers are algebras whose multiplications depend on $x \in X$ smoothly. A *differential connection* ∇ on E is a smooth connection such that for any vector field v on X, we have:

$$\nabla_v(\sigma_1\sigma_2) = \sigma_1\nabla_v(\sigma_2) + \nabla_v(\sigma_1)\sigma_2$$

for any sections $\sigma_1, \sigma_2 \in \Gamma(X, E)$.

Surprisingly, existence of such connections is a sufficient condition for the fiber algebras to be isomorphic. For a necessary condition, one needs a stronger assumption than just having isomorphic fiber algebras. For a detailed discussion on connections, see Chern *et al.* (1999). We will formalize these statements in the next two propositions.

Proposition 3. *If E has a differential connection ∇, then the fiber algebras of $E \twoheadrightarrow X$ are all isomorphic.*

Proof. Assume E has a differential connection ∇. Let $x, y \in X$ and let $I \xrightarrow{\gamma} X$ be a (piecewise) smooth path in X with $\gamma(0) = x$ and $\gamma(1) = y$. Using the connection ∇, we have a parallel transport map that is a linear isomorphism:

$$\Phi(\gamma)_x^y\colon E_x \longrightarrow E_y$$

Thus, all we have to show is that $\Phi(\gamma)_x^y$ is multiplicative. Given $b_1, b_2 \in E_x$, there are unique smooth sections σ_1 and σ_2 of E along γ such that $\nabla_{\vec{\gamma}}\sigma_1 = \nabla_{\vec{\gamma}}\sigma_2 = 0$ and $\sigma_1(x) = b_1$ and $\sigma_2(x) = b_2$. Here, $\vec{\gamma}$ denotes the smooth tangent vector field of γ. Note that the product $\sigma_1\sigma_2$ is the unique smooth section of $E \twoheadrightarrow X$ along γ such that $(\sigma_1\sigma_2)(x) = \sigma_1(x)\sigma_2(x) = b_1b_2$ and:

$$\nabla_{\vec{\gamma}}(\sigma_1\sigma_2) = \sigma_1\nabla_{\vec{\gamma}}(\sigma_2) + \nabla_{\vec{\gamma}}(\sigma_1)\sigma_2 = 0$$

Thus, by definition of the parallel transport map $\Phi(\gamma)_x^y$, we have:

$$\Phi(\gamma)_x^y(b_1b_2) = (\sigma_1\sigma_2)(y) = \sigma_1(y)\sigma_2(y) = \Phi(\gamma)_x^y(b_1)\Phi(\gamma)_x^y(b_2)$$

which shows that $\Phi(\gamma)_x^y$ is multiplicative. ■

A strong converse of the above proposition, where the isomorphisms among fibers satisfy some coherence conditions, holds. By a *coherent collection*:

$$\mathcal{P} = \left\{\Phi(\gamma)_x^y\colon E_x \longrightarrow E_y \,\middle|\, \forall\, x, y \in X,\ I \xrightarrow{\gamma} X\ smooth\right\}$$

of isomorphisms among fibers of $E \twoheadrightarrow X$, we mean a collection satisfying:

a) $\Phi(\gamma)_x^x = id$,
b) $\Phi(\gamma)_u^y \circ \Phi(\gamma)_x^u = \Phi(\gamma)_x^y$, and
c) Φ depends smoothly on γ, y and x.

We then have the following proposition.

Proposition 4. *A coherent collection $\mathcal{P}$ of algebra isomorphisms on fibers of $E \twoheadrightarrow X$ gives a differential connection ∇ on E.*

Proof. Using the collection $\mathcal{P}$, we can immediately write an infinitessimal connection ∇ as follows: for any vector V on X we have:

$$\nabla_v(\sigma) = \lim_{t\to 0}\frac{\Phi(\gamma)_{\gamma(t)}^x\sigma\big(\gamma(t)\big) - \sigma(x)}{t} = \frac{d}{dx}\bigg|_{t=0}\Phi(\gamma)_{\gamma(t)}^x\sigma\big(\gamma(t)\big)$$

for any $\sigma \in B, x = \gamma(0)$, and $\nu = \gamma'(0)$. That ∇ is a differential connection follows from the multiplicativity of $\Phi(\gamma)_x^y$ and the Leibniz property of $\left.\frac{d}{dx}\right|_{t=0}$. ■

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