Philippine Journal of Science
145 (4): 385-393, December 2016
ISSN 0031 - 7683

Mass-Dependent Arrival Time Density of a Ballistic Particle at the Turning Point

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The arrival time density of a ballistic particle (projected towards the turning point) is determined using a general form of Kijowski's distribution. For given parameter values, two asymmetric peaks of the arrival time density are obtained, each arising respectively from the amplitudes for positive and negative momenta of the particle. These peaks represent the most probable arrival times before (for positive momentum) and after (for negative momentum) the classical arrival time. The features of the arrival time density such as its peaks and amplitude are shown to vary with the particle's mass for a given initial position uncertainty.

Key words: arrival time density, ballistic particle, Kijowski's distribution, quantum mechanics,

INTRODUCTION

Quantum mechanics is a quantitative description of the wave-particle behavior of microscopic objects. Its postulates and conclusions are expressed in probabilistic terms. If one measures an observable Q for a given state Ψ of that object, quantum mechanics predicts the probability density ∏(Q) of measuring particular values of Q, and talks about the expectation value, correlations with other observables, and standard deviation of Q. It does not and cannot predict the outcome of a single measurement of Q. However, if one performs the same measurement repeatedly, using an ensemble of objects with identical states Ψ, the statistical picture that emerges must be consistent with the probability density ∏(Q).

This article poses the question: when does a quantum particle arrive at a given point? Consider a detector at some height $z = z_f$ and a ballistic particle confined along the vertical z-axis, prepared in some initial state and represented by a wave function in coordinate space, i.e. position amplitude $\Psi(z, t_i)$. The particle is measured by the detector at some later time $t = t_f$. The arrival time at the detector is defined as $T = t_f - t_i$, but because of quantum indeterminacy we expect to find a probability density of different arrival times at the same arrival point z_f which we refer to as the arrival time density ∏(T). The probability that the particle arrives at the detector in the time interval $t_1 < T < t_2$ is

$$P(t_1 < T < t_2) = \int_{t_1}^{t_2} \Pi(T)\, dT$$

Known results concerning the arrival time density ∏(T) are discussed in various review articles: Egusquiza et al. 2002; Sokolovski 2008; Galapon 2009b, Ruschhaupt et al. 2009; Wynands 2009. For a ballistic particle, why would it be significant?

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METHODS

Arrival Time and the Weak Equivalence Principle

The arrival time density $\prod(T)$ can re-examine a principle of physics called geometric weak equivalence. This principle states that the mass of a test particle under constant gravity is irrelevant in determining its future state of motion. There are alternate versions of the weak equivalence principle in relativity physics: (a) gravitational and inertial masses are equal, (b) the laws of motion of freely falling particles take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation, and (c) any two test bodies must fall with the same acceleration in a given external gravitational field (Weinberg 1972; Misner et al. 1973). Geometric weak equivalence and its variants express the same physical meaning when applied to macroscopic objects, and are the starting points of the general theory of relativity.

Do these principles apply for microscopic particles? Testing the validity of the general theory of relativity in microscopic physics is currently an active area of research (Dimopoulos 2007). Fray et al. compared the gravitational acceleration g of the two atomic Rubidium isotopes ^{85}Rb and ^{87}Rb, obtaining a difference of $\Delta g/g = (1.2 \pm 1.7) \times 10^{-7}$. Peters et al. (1999) used an atom interferometer to achieve an absolute uncertainty of $\Delta g/g \approx 3\times10^{-9}$.

From a theoretical point of view an interesting analysis on geometric weak equivalence was made by Davies (2004). A particle projected upward can tunnel into the classically forbidden region above the turning point z_{TP}. This tunneling depth depends on the mass m so a mass-dependent quantum delay is expected. At present there is no universally accepted notion of tunneling time, i.e. the time spent by the particle inside the classically forbidden region (Landauer and Martin 1994). To measure this quantum delay, Davies used a Peres clock (Peres 1980). A Peres clock is an equally microscopic object, weakly coupled to the tunneling particle, with one degree of freedom: a rotor in an initial state with a well-defined pointer angle, running only when the particle traverses the space between two points of interest.

Davies calculated the quantum delay at the turning point. It is approximately equal to $0.5(\hbar/mg^2)^{1/3}$ introducing a mass-dependent quantum correction to the classical arrival time of the tunneling particle. Such a delay represented a violation of geometric weak equivalence. This implies that geometric weak equivalence is not the same as the alternate formulations of weak equivalence – at least not microscopically.

The theoretical limitations of geometric weak equivalence in microscopic physics were also pointed out in previous work using entirely different approaches on the same problem: determining the arrival time behavior of microscopic ballistic particles released from a given initial mean height z_0. This is the quantum version of the famous but apocryphal Pisa experiment by Galileo. Viola and Onofrio (1997) used stochastic mechanics to derive mass-dependent quantum fluctuations around the average arrival time. Ali et al. (2006) computed a mass-dependent mean arrival time (using the quantum probability current). Villanueva and Galapon (2010) obtained mass-dependent arrival time densities using two approaches: crossing states and generalized crossing states, the latter arising from a consideration of a time-of-arrival operator in the interacting case. In all three papers the quantum deviations from the classical arrival time showed up, but became negligible for large mass so that geometric weak equivalence was recovered.

This paper extends the previous work on the arrival time density of a particle influenced by a linear potential such as a constant gravitational field (Villanueva and Galapon 2010). Investigation of ballistic particles is significant because freely falling atoms and atomic fountains have been realized experimentally, and a theoretical prediction can be tested with actual arrival time statistics. The author investigated the time evolution of a minimum uncertainty wave packet in the vicinity of the turning point, calculated its arrival time density, and determined that these properties exhibit mass dependence.

Kijowski's Distribution

Using a standard quantum mechanical approach to the time-of-arrival problem, Egusquiza and co-workers obtained Kijowski's distribution (Egusquiza et al. 2002),

$$\Pi(T) = \Pi_+(T) + \Pi_-(T) \qquad (1)$$

where

$$\Pi_+(T) = \left| \int_0^\infty dp \sqrt{\frac{p}{2\pi\hbar m}}\, e^{\frac{ipz_f}{\hbar}} \phi(p,T) \right|^2 \qquad (2)$$

$$\Pi_-(T) = \left| \int_{-\infty}^0 dp \sqrt{\frac{-p}{2\pi\hbar m}}\, e^{\frac{ipz_f}{\hbar}} \phi(p,T) \right|^2 \qquad (3)$$

Kijowski's distribution is the arrival time density of a free particle of mass m at the given arrival point $z = z_f$ where $\varphi(p,T)$ is the wave function in momentum space, i.e. momentum amplitude of the free particle at time t=T. Recall that the momentum amplitude $\varphi(p,T)$ and the position amplitude $\Psi(z,T)$ are Fourier transforms of each other. Kijowski's distribution depends the arrival time T, the arrival point z_f and the momentum amplitude φ(p,T). In a more compact notation, this paper will adopt

$$\Pi(T, z_f, \varphi(p,T)) \rightarrow \Pi(T)$$

Kijowski's distribution is an *ideal* arrival time density, i.e. the arrival time density is calculated without any reference to the manner of particle detection which is typically a highly specific interaction between the arriving particle and the detector, which absorbs the particle. Kijowski's distribution also does not require an internal degree of freedom as a quantum clock (such as a Peres clock). Another attractive feature of Kijowski's distribution is covariance with respect to time translations so that the arrivals predicted for a given fixed instant are independent of the choice made for the initial time. In other words, $\prod(T, z_f, \varphi(p,0)) = \prod(T-T', z_f, \varphi(p, T'))$. Known results concerning the arrival time density ∏(T) are discussed in various review articles: Egusquiza et al. 2002; Sokolovski 2008; Galapon 2009b, Ruschhaupt et al. 2009; Wynands 2009.

The two terms on the RHS of expression (1) are interpreted as contributions to the arrival time density due to the momentum amplitude φ(p,T) for positive and negative momenta, respectively. Since classical particles with positive and negative momenta along the vertical z-axis can be alternatively identified as upward-moving and downward-moving particles respectively, a secondary physical interpretation has crept into the theory; namely that $\prod_+(T)$ is in addition to be also understood as the relative probability density of detecting the particle at time t = T from below of the arrival point z_f, and $\prod_-(T)$ the relative probability density of detection at time t = T from above z_f.

Two works (Baute et al. 2000; Baute et al. 2001) suggested a straightforward extension of Kijowski's distribution when there is an interaction potential present by the replacement φ(p,T) → Φ(p,T) in expressions (2) and (3), where Φ(p,T) is the momentum amplitude of the particle under the influence of the interaction potential V(z) at time t = T. In this case, the arrival time density does not distinguish between first and subsequent arrivals and for this reason ∏(T) may not normalizable (but we can still compare relative arrival time probability densities). For instance, if the motion is periodic such as that of a harmonic oscillator, repeated arrivals at the same point at different times may occur.

Until quite recently, people generally agree on Kijowski's distribution as the appropriate arrival time density for free particles (without internal degrees of freedom such as spin), but there is no consensus yet on a general theory in the interacting case (Muga and Leavens 2000). More elaborate theories entail the construction of a self-adjoint time-of-arrival operator defined on some interval on the real line (Galapon 2004; Galapon and Villanueva 2008; Galapon 2009a), or operator normalization with a highly specific detector model (Hegerfeldt et al. 2003; Hegerfeldt et al. 2004).

Sombillo and Galapon (Sombillo and Galapon 2016) recently argued that the Kijowski distribution -- interpreted as the arrival time density of a free particle -- cannot reproduce both the temporal and the spatial profile of the modulus squared of the time-evolved wave function for an arbitrary initial state. In particular using a specific wave function given to be zero at a certain point x_0 for all values of time, they showed that Kijowski's distribution at x_0 gives a non-vanishing arrival time probability if the wave function contains both positive and negative momentum components. In other words, Kijowski's distribution cannot be regarded as a tenable arrival time distribution if the free particle contains both positive and negative momentum components. Inasmuch as the current object of interest is a particle subject to gravity, the weight of their objections must still be carefully considered.

At this point a clarification of physical content of $\prod_+(T)$ and $\prod_-(T)$ seems a necessary preliminary step. Can $\prod_+(T)$ and $\prod_-(T)$ retain the additional secondary interpretation of being relative arrival time densities for the particle coming from below and above z_f, respectively, at time t = T?

First, note that the positive momentum contribution $\prod_+(T)$ arise from the position amplitude Ψ(z, T) from both below as well as above the turning point z_{TP}. Consider the complex-valued momentum amplitudes

$$M_{CA}(p,T) = \int_{-\infty}^{zTP} dz \sqrt{\frac{1}{2\pi\hbar}}\, e^{\frac{ipz}{\hbar}}\, \Psi(z,T) \quad (4)$$

$$M_{CF}(p,T) = \int_{zTP}^{\infty} dz \sqrt{\frac{1}{2\pi\hbar}}\, e^{\frac{ipz}{\hbar}}\, \Psi(z,T) \quad (5)$$

where $M_{CF}(p, T)$ is the momentum amplitude due to the position amplitude Ψ(z,T) within the classically forbidden region, and $M_{CA}(p,T)$ is the corresponding momentum amplitude from the classically allowed region, with sum $M_{CF}(p,T) + M_{CA}(p,T) = \Phi(p,T)$. Second, the author defines the complex-valued arrival amplitudes

$$A_{CA}{}^{+}(T) = \int_{0}^{\infty} dp \sqrt{\frac{p}{2\pi\hbar m}}\, e^{\frac{ipz_f}{\hbar}}\, M_{CA}(p,T) \quad (6)$$

$$A_{CF}{}^{+}(T) = \int_{0}^{\infty} dp \sqrt{\frac{p}{2\pi\hbar m}}\, e^{\frac{ipz_f}{\hbar}} M_{CF}(p,T) \quad (7)$$

which are amplitude contributions to the arrival time density (2) in the classically allowed ($z < z_{TP}$) and classically forbidden regions ($z > z_{TP}$) respectively. It seems natural to associate with A^{+}_{CA} the amplitude for arrival from below, and A^{+}_{CF} the amplitude for arrival from above the turning point z_{TP}. The arrival amplitudes A^{+}_{CA} and A^{+}_{CF} satisfy $\prod_{+}(T) = |A^{+}_{CA} + A^{+}_{CF}|^2 = |A^{+}_{CA}|^2 + 2\mathrm{Re}((A^{+}_{CF})^{*}A^{+}_{CA}) + |A^{+}_{CF}|^2$ implying that $\prod_{+}(T)$ is actually a result of the interference between arrival amplitudes A^{+}_{CA} and A^{+}_{CF}. Recall that $|z|^2 = z^{*}z$ for any complex value z, where z* is the conjugate of z, and Re(z) is the real part of z.

Hence $\prod_{+}(T)$ arises from contributions of the position amplitude Ψ(z, T) from below as well as from above the turning point z_{TP} plus an interference term $2\mathrm{Re}((A^{+}_{CF})^{*}A^{+}_{CA})$ which is not guaranteed to be always positive. Therefore we cannot interpret $\prod_{+}(T)$ as exclusively the relative arrival time density from below the turning point. In other words, $\prod_{+}(T)$ is not related to the time exclusively spent by the particle in the classically allowed region $z<z_{TP}$.

A similar negative conclusion can be made for $\prod_{-}(T)$ (considered as the relative arrival density from above the turning point) by defining analogous arrival time amplitudes $A^{-}_{CF}(T)$ and $A^{-}_{CA}(T)$ for negative momenta so that $\prod_{-}(T) = |A^{-}_{CA} + A^{-}_{CF}|^2 = |A^{-}_{CA}|^2 + 2\mathrm{Re}((A^{-}_{CF})^{*}A^{-}_{CA}) + |A^{-}_{CF}|^2$. Hence $\prod_{-}(T)$ cannot be associated with the time spent by the particle inside the classically forbidden region ($z>z_{TP}$), i.e. the tunneling time.

Minimum Uncertainty Wave Packet in a Constant Gravitational Field

Aside from the important question of whether or not geometric weak equivalence is valid on the microscopic level, a theoretical investigation of the arrival time behavior of particles in a constant gravitational field can be compared with the arrival time statistics of an atomic fountain experiment. In an atomic fountain, atoms are first stored and cooled in an optical trap. After preparation the atoms are given an initial push upwards by a laser pulse. Behaving as ballistic particles they arrive at the turning point z_{TP} and later fall down into a detection region.

The standard quantum mechanical treatment of a ballistic particle (the internal structure of which such as spin is ignored) can be conveniently done in terms of the Gaussian momentum amplitude Φ(p, t) (Robinett 1996; Robinett 2006)

$$\Phi(p,t) = \Phi_0(p + mgt)\exp\left(-\frac{itp^2}{2m\hbar} - \frac{ipgt^2}{2\hbar} - \frac{img^2t^3}{6\hbar}\right) \quad (8)$$

where Φ_0 is initial momentum amplitude

$$\Phi_0(p) = \sqrt{\frac{\sqrt{8\pi\sigma_0{}^2}}{2\pi}}\exp\left(-(p-p_0)^2\frac{\sigma_0{}^2}{\hbar^2} - i(p-p_0)\frac{z_0}{\hbar}\right) \quad (9)$$

and m is the particle mass, g is the constant acceleration due to gravity, σ_0 is the initial position uncertainty, p_0 is the initial mean momentum, and z_0 is the initial mean position. The momentum density $|\Phi(p,t)|^2 = |\Phi_0(p+mgt)|^2$ does not spread out but translates to the left over time, and the mean momentum becomes zero at t_{CL} which is the arrival time for a classical particle at the turning point z_{TP}. The initial momentum amplitude (9) is the Fourier transform of the minimum uncertainty wave packet Ψ(x,0) with initial position uncertainty σ_0, and initial mean position z_0. The momentum amplitude Φ(p,t) for negative momentum is exponentially small before $t = t_1$ such that $\sigma_p = p_0 - mgt_1$ where $\sigma_p = \hbar / 2\sigma_0$ is the initial momentum uncertainty. This is $t_1 = t_{CL} - \tau$ where $\tau = \hbar / 2mg\sigma_0$. For the same consideration the momentum amplitude Φ(p,t) for positive momentum is exponentially small after $t = t_2$ where $t_2 = t_{CL} + \tau$. Therefore 2τ is a time scale where the particle, initially projected upwards, changes momentum.

RESULTS AND DISCUSSION

Parameters

Experiments with atomic fountains typically use classical time-of-flight calculations. Under what conditions is a classical description justified? The spreading of the position density $|\Psi(z,t)|^2$ where Ψ(z,t) is a Gaussian wave packet is a peculiar feature of quantum mechanics, and for a ballistic particle this spreading is described by the position uncertainty $\Delta z = \sigma_0 (1 + \hbar^2t^2/4m^2\sigma_0{}^4)^{1/2}$ (Robinett 1996). In terms of a 'coherence time' (Robinett 2006) where $t_0 = 2m\sigma_0{}^2/\hbar$, the spreading is negligible in the time scale that is small relative to t_0. Hence if t_{CL} is small compared to the coherence time, a classical time-of-flight calculation should be adequate. In this paper, the author chose parameters (shown in Table 1) such that t_{CL} is not small relative to t_0. As seen in Figure 1, the spreading of the position density $|\Psi(z,t)|^2$ is a dominant feature in this regime.

According to Ehrenfest's Theorem, the position expectation value $\langle z(t) \rangle$ of the ballistic particle satisfies the classical equation of motion with initial velocity v_0 and initial position z_0. Therefore, the initial mean momentum is chosen to be positive $p_0 > 0$ so the position expectation

Table 1. Parameters of the Ballistic Particle.

Parameter	Physical significance	Atomic units (a.u.)	Metric (MKS)
z_0	initial mean position	0	0
z_{TP}	classical turning point	1000	5.292×10^{-8} m
σ_0	initial position uncertainty	10	5.292×10^{-9} m
t_{CL}	classical arrival time at z_{TP}	4.292×10^{12}	1.039×10^{-4} s
t_0	spreading time	$2m\sigma_0^2/\hbar$	$2m\sigma_0^2/\hbar$
2τ	time scale for momentum change	$\hbar /2mg\sigma_0$	$\hbar /2mg\sigma_0$
t_1	Momentum amplitude $\varphi(p,t)$ for positive momentum exponentially small for $t>t_1$	$t_{CL} - \tau$	$t_{CL} - \tau$
t_2	Momentum amplitude $\varphi(p,t)$ for negative momentum exponentially small for $t<t_2$	$t_{CL} + \tau$	$t_{CL} + \tau$
v_0	initial mean velocity	4.657×10^{-10}	1.019×10^{-3} m/s
p_0	initial mean momentum	mv_0	mv_0

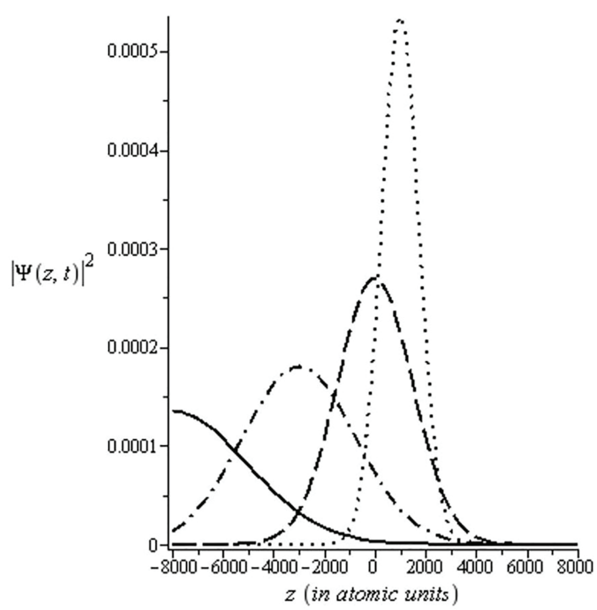

Figure 1. Position density $|\Psi(z,t)|^2$ of a ballistic particle at different times: $|\Psi(z, t_{CL})|^2$ (dot), $|\Psi(z, 2t_{CL})|^2$ (dash), $|\Psi(z, 3t_{CL})|^2$ (dash dot), and $|\Psi(z, 4t_{CL})|^2$ (solid). The particle's mass is $m = 10m_{Cs}$ where $m_{Cs} = 2.903 \times 10^6$ in atomic units. Length z is in atomic units (a.u.). The turning point is $z_{TP} = 1000$ a.u. The initial position density $|\Psi(z,0)|^2$ is not included.

value $< z(t) >$ initially rises from z_0, reaches the classical turning point z_{TP} at time $t = t_{CL}$ and subsequently accelerates downward. The initial mean velocity is $v_0 = p_0/m = 0.1019$ cm/sec, and the particle's mass is $m = 10m_{Cs}$ where $m_{Cs} = 2.903 \times 10^6$ (in atomic units) is the mass of the cesium atom. The chosen initial velocity v_0 is comparable to those of cold atoms found in atomic fountain experiments. The classical arrival time at the turning point z_{TP} is $t_{CL} = v_0/g = 0.0001039$ sec. As a consequence of Ehrenfest's theorem, every ballistic particle that has the same initial mean velocity v_0 and initial mean position z_0 will have identical position expectation values $< z(t) >$ at any time $t>0$. In particular, $< z(t_{CL}) >= z_{TP}$. Since investigating the dependence of the arrival time density with the particle's mass is the object of this paper, the author compared the arrival time densities of ballistic particles that differ in mass m_1 and m_2 *and* initial mean momenta $(p_0)_1$ and $(p_0)_2$ respectively such that $(p_0)_1/m_1 = (p_0)_2/m_2 = v_0 = 0.1019$ cm/sec (for the same initial mean position $z_0 = 0$).

Wave Packet Behavior

For $t > 0$ the position density $|\Psi(z,t)|^2$ spreads out rapidly from the initial position density $|\Psi(z,0)|^2$ (a very sharp and narrow Gaussian if shown in the scale of figure 1). This spreading of the wave packet is independent of the direction of the initial velocity v_0, but depends on the particle's mass (larger masses have smaller spreading). Wave packet spreading is caused by the interference among the momentum components of the wave function. The wave packet contains a range of different momenta (determined by the momentum uncertainty Δp) and corresponding angular frequencies $\omega=p^2/2m\hbar$. Thus each p-component oscillates as $\exp(-i\omega t)$. The time-dependent phase relations between the p-components correspondingly change the regions of constructive and destructive interference which shape the wave packet. Around time $t = t_{CL}$ there is noticeable penetration of the position density $|\Psi(z,t)|^2$ into the classically forbidden region ($z>1000$ atomic units) as shown in Figure 1, up to approximately time $t = 4t_{CL}$. The tunneling probability at time t is

$$P_{tunnel}(t) = \int_{z_{TP}}^{\infty} dz\ |\Psi(z,t)|^2 \qquad (10)$$

and according to Figure 1 there is a significant probability for tunneling by the particle in the time interval $0 < t < 4t_{CL}$. Figure 2 shows the mass dependence of the tunneling probability in the same time interval $0 < t < 4t_{CL}$. For larger masses ($m>10m_{Cs}$), the spreading is less pronounced, mitigating the penetration into the classically forbidden region, and therefore tunneling occurs in a shorter time interval.

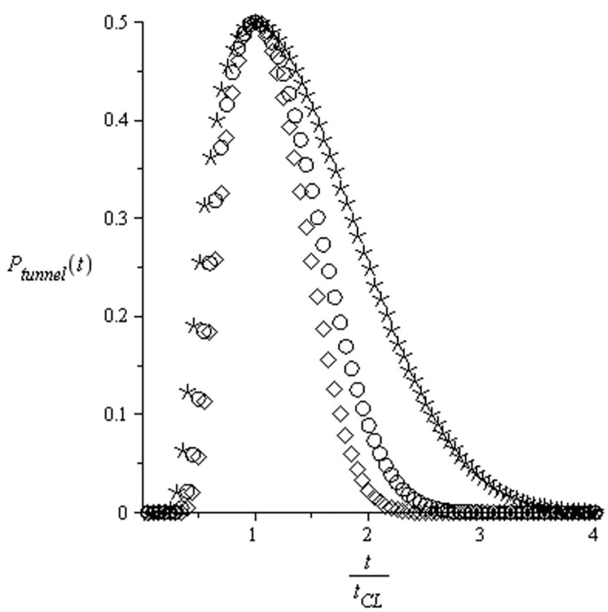

Figure 2. Ratio t/t_{CL} vs. tunneling probability P_{tunnel} (t) for different particle masses $m = 10m_{Cs}$ (asterisk), $m = 20m_{Cs}$ (circle) and $m = 30m_{Cs}$ (diamond) where $m_{Cs} = 2.903 \times 10^6$ (in atomic units).

For the given mass $m = 10m_{Cs}$, and in terms of t_{CL}, we have $t_1 = 0.632t_{CL}$ and $t_2 = 1.38t_{CL}$. This implies that, in figure 2 the tunneling probability in the time interval $0 < t < 0.632t_{CL}$ is attributable to the positive momentum components of the wave packet, while in the time interval $0.632t_{CL} < t < 1.38t_{CL}$ the tunneling probability is due to both positive and negative components, and finally in the time interval $t > 1.38t_{CL}$ the tunneling probability is due to the negative momentum components. Analogous conclusions can be made for different masses for appropriate values of t_1 and t_2.

For how long is particle detection at the turning point significant? Consider

$$P_{\sigma_0}(t) = \int_{z_{TP}-\sigma_0}^{z_{TP}+\sigma_0} dz\ |\Psi(z,t)|^2 \qquad (11)$$

representing the particle's position probability in the vicinity of the detector at $z = z_{TP}$ (for a given σ_0) , i.e. the detection probability. Figure 3 shows a comparison of the detection probability $P\sigma_0(t)$ for the same set of particles considered in figure 2. The spread of the detection probability $P\sigma_0(t)$ is approximately the time interval where particle detection at the turning point is significant. This time interval is comparatively short for larger masses. The plots of $P_{tunnel}(t)$ and $P\sigma_0(t)$ are asymmetric for small masses, and this asymmetry can be attributed to wave packet spreading. The position density $|\Psi(z,t)|^2$ is broader as it descends from the turning point, so the "tail" of $|\Psi(z,t)|^2$ will pass by the region of interest for a longer time, compared to the position density $|\Psi(z,t)|^2$ as it ascends to the turning point.

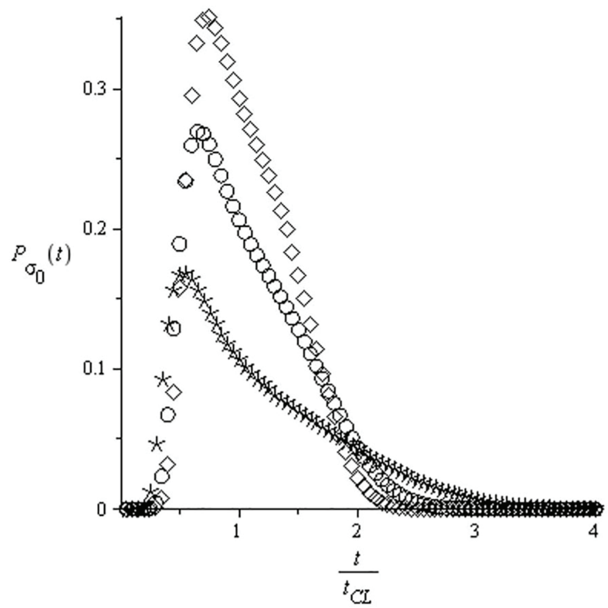

Figure 3. Ratio t/t_{CL} vs. detection probability $P_{\sigma 0}$ (t) for different particle masses $m = 10m_{Cs}$ (asterisk), $m = 20m_{Cs}$ (circle) and $m = 30m_{Cs}$ (diamond) where $m_{Cs} = 2.903 \times 10^6$ (in atomic units).

Arrival Time Distribution

The arrival time density of a particle projected upwards is calculated using expressions (1), (2) and (3) where the momentum amplitude $\varphi(p,T) \rightarrow \Phi(p, t)$ is given by expression (8). The calculation of the integral in expression (2) can be done exactly, using the identity

$$\int_0^\infty dp\, p^{\upsilon-1} e^{-\beta p^2-\gamma p} = (2\beta)^{-\upsilon/2}\,\Gamma(\upsilon)\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-\upsilon}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \qquad (12)$$

provided Re(β) > 0 and Re(ν) >0, where $D_\nu(z)$ is the parabolic cylinder function of order ν and Γ(z) is the Gamma function (Gradshteyn 2007). In this case ν=3/2 and

$$\beta = \frac{\sigma_0{}^2}{\hbar^2} + \frac{iT}{2m} \qquad (13)$$

$$\gamma = \frac{2\sigma_0{}^2}{\hbar^2}(mgT - p_0) + \frac{i}{\hbar}\left(z_0 - z_f + \frac{gT^2}{2}\right) \qquad (14)$$

The calculation of expression (3) is similar, differing only in the final analytic result from the sign of the argument of $D_{-3/2}(z)$.

The plot of $\prod\pm(T)$ at the turning point is provided in Figure 4 for the same set of masses considered in Figures 1 and 2. Since the physical interpretation of $\prod\pm(T)$ as arrival time densities is being questioned if both positive and negative momentum components are significant (Sombillo and Galapon 2016), we restrict our results to time $T < t_1$ for $\prod_+(T)$. Recall that the momentum amplitude Φ(p,T) for negative momentum become exponentially small before time $t_1 = t_{CL} - \tau$ where $\tau = \hbar / 2mg\sigma_0$. Note t_1 increases as the particle's mass increases, and it approaches t_{CL}. If we consider five different masses ($10m_{Cs}$, $15m_{Cs}$, $20m_{Cs}$, $25m_{Cs}$ and $30m_{Cs}$) the minimum t_1 is t_{plus} = 0.0000657 sec equivalent to $t_{plus} = 0.632\ t_{CL}$. In Figure 4A we compare the

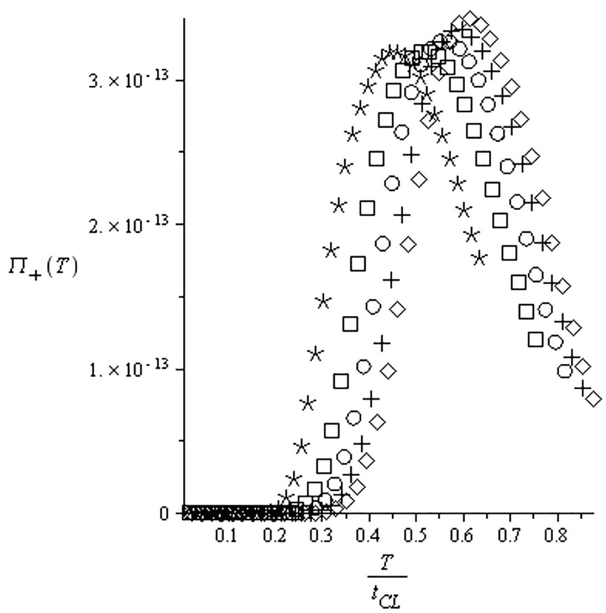

Figure 4A. $\prod_+(T)$ for different particle masses $m = 10m_{Cs}$ (asterisk), $m = 15m_{Cs}$ (box), $m = 20m_{Cs}$ (circle), $m = 25m_{Cs}$ (cross) and $m = 30m_{Cs}$ (diamond) where $m_{Cs} = 2.903 \times 10^6$ (in atomic units).

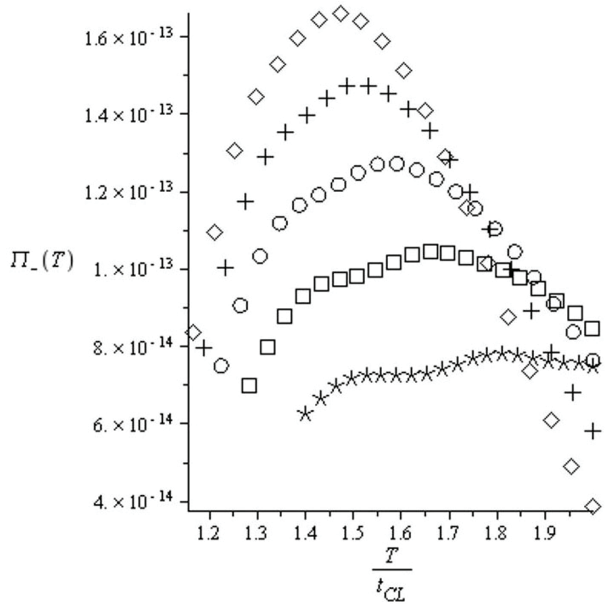

Figure 4B. $\prod_-(T)$ for different particle masses $m = 10m_{Cs}$ (asterisk), $m = 15m_{Cs}$ (box), $m = 20m_{Cs}$ (circle), $m = 25m_{Cs}$ (cross) and $m = 30m_{Cs}$ (diamond) where $m_{Cs} = 2.903 \times 10^6$ (in atomic units).

positive Kijowski's distributions $\prod_+(T)$ for the same five masses which we interpret as the arrival time densities in the time interval $0 < T < 0.632\ t_{CL}$. The peaks of the arrival time densities vary with the particle's mass and appear to shift to the right (approaching the classical arrival time t_{CL}) for increasing mass. Similarly, for $\prod_-(T)$ we restrict our results to time $T > t_2$ where the momentum amplitude Φ(p,T) for positive momentum become exponentially small after time $t_2 = t_{CL} + \tau$. Note t_2 decreases as the particle's mass increases, and it approaches t_{CL}. For the same five masses previously considered ($10m_{Cs}$, $15m_{Cs}$, $20m_{Cs}$, $25m_{Cs}$ and $30m_{Cs}$) the maximum t_2 is t_{minus} = 0.000142 sec equivalent to $t_{minus} = 1.38\ t_{CL}$. In figure 4B we compare the negative Kijowski's distributions $\prod_-(T)$ for the same five masses which we interpret as the arrival time densities in the time interval $T > 1.38\ t_{CL}$. The peaks of the arrival time densities again vary with the particle's mass and appear to shift to the left for increasing mass.

Let $T = T^\pm_{peak}$ be the time where $\prod\pm(T)$ attains its maximum. The effect of the particle's mass on the most probable arrival time $T^\pm_{peak}$ is shown in Figure 5, where the particle mass ratio m/m_{Cs} vs. the ratio of $T^\pm_{peak}$ with the classical arrival time t_{CL} is plotted in the mass range $10m_{Cs} < m < 30m_{Cs}$. Quantum deviations of the most probable arrival time from the classical arrival time are apparent, and more pronounced for low particle mass m. For increasing mass, the most probable arrival time appears to slowly approach t_{CL}.

In Figure 5 the most probable arrival time T^{+}_{peak} due to momentum amplitude Φ(p,T) for positive momentum is consistently less than the classical arrival time. This signifies an early arrival time, attributed to the momentum amplitude that produce the upper positive end of the momentum distribution $|\Phi(p, T)|^2$. As the penetration of the classically forbidden region at the time interval $0 < t < t_1$ is due to these positive momentum components, *a tunneling particle may therefore arrive at the turning point earlier than the classical arrival time*. For increasing particle mass m, T^{+}_{peak} approaches the classical arrival time. Note these positive momentum components generally come from position amplitudes from below as well as above the turning point. Therefore T^{+}_{peak} is not exclusively the time spent by the particle in the classically forbidden region, i.e. it is not a tunneling time.

A quantum delay was also obtained given by $T^{-}_{peak} - t_{CL}$ where T^{-}_{peak} is due to the momentum amplitude for negative momenta. From the previous discussion, this is not the arrival time of the particle from above the turning point. Hence $T^{-}_{peak} - t_{CL}$ is not exclusively the time spent by the particle in the classically forbidden region, i.e. $T^{-}_{peak} - t_{CL}$ is also not a tunneling time.

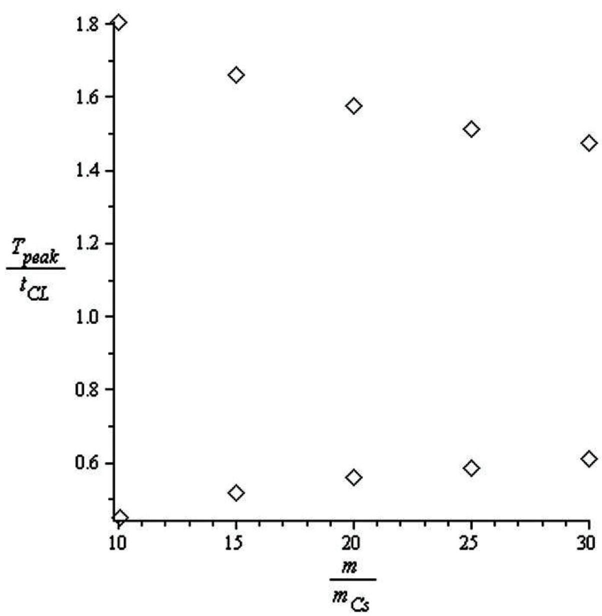

Figure 5. Particle mass ratio m/m_{Cs} vs. $T^{\pm}_{peak}/t_{CL}$. The mass range is $10m_{Cs} < m < 30m_{Cs}$ where $m_{Cs} = 2.903 \times 10^6$ (in atomic units). T^{+}_{peak}/t_{CL} (circles) T^{-}_{peak}/t_{CL} (diamonds).

CONCLUSION

The results in this paper show that, contrary to the classical case, ballistic particles with the same initial mean velocity v_0 and initial mean position z_0 behave in a manifestly mass-dependent manner. The spreading of the wave packet is more pronounced for less massive particles. This in turn affects experimentally measurable quantities like the tunneling probability $P_{tunnel}(t)$, the detection probability at the turning point $P_{\sigma 0}(t)$, and the arrival time density Π(T).

For given parameter values, two asymmetric peaks of the arrival time density are obtained, each arising respectively from the amplitudes for positive and negative momenta of the particle. These mass-dependent peaks represent the most probable arrival times before and after the classical arrival time. The earlier arrival time peak is due to the positive momentum components of the wave packet. This implies the surprising result that a tunneling particle may arrive earlier than the classical arrival time. These features of the arrival time density are shown to vary with the particle's mass for a given initial position uncertainty, while remaining consistent with the quantum dynamics of a ballistic particle in the vicinity of the turning point.

With these results, the author concludes that geometric weak equivalence is not generally valid in a microscopic context. The variation of the tunneling probability, detection probability, and arrival time density with the particle's mass suggests that novel physics may possibly be observed among quantum particles under constant gravity.

ACKNOWLEDGEMENTS

The author gratefully acknowledges useful discussions with Dr. E. A. Galapon.

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