Parameter Estimation on a Model of a Heat-Conducting Rod: Mathematical Analysis and Numerical Computations

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We model the heat flow of an experiment involving a metallic rod with a heat source at one end. By fitting the mathematical model to experimental data, we were able to estimate the ratios of the rod's thermal conductivity, heat capacity, and flux of the heat source over the heat transfer coefficient. Temperature measurements were obtained via thermocouples located at seven points along the length of the rod. Both steady-state and time-dependent data were used in the parameter estimation, and for the time-dependent case, a numerical algorithm for approximating the partial differential equation was implemented. Since the model featuring realistic boundary conditions does not have an explicit solution, we showed the well-posedness of the model using mathematical analysis. The convergence of the finite-dimensional approximations of the model to the infinite-dimensional solution was also discussed. To solve the optimization problem arising from the parameter estimation problem, we used the least squares and genetic algorithms. Our numerical results indicate that these two optimization algorithms converged to the same solution.

Key Words: parameter estimation, heat propagation model, numerical optimization algorithms, mathematical modeling, well-posedness

INTRODUCTION

In this paper, we use the one-dimensional heat equation to model an actual experiment consisting of a metallic rod with a heat source at one end, and with thermocouples measuring the transient and steady-state temperature. The partial differential equation was formulated to incorporate heat loss along the length of the rod and to satisfy realistic boundary conditions: heat flux input at one end and heat loss at the other end. The resulting equation has no analytic solution due to the incorporation of realistic boundary conditions, hence numerical methods must be implemented to approximate the solution. Thus, the convergence of the numerical approximations must be established and to accomplish this, we first show the existence and uniqueness of the partial differential equation posed in an infinite dimensional Hilbert space, then we cite existing results that guarantee the convergence of the numerical approximations to the infinite-dimensional solution.

By fitting the model parameters to temperature measurements, we are able to estimate the ratios of the metallic rod's heat capacity, the heat transfer coefficient, and the flux of the heat source with the rod's thermal conductivity. The parameter estimation problem ultimately leads to an optimization problem which we solved using two different optimization algorithms: a Newton type (gradient based) algorithm and a genetic algorithm. Our results show that these two fundamentally different

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algorithms can both solve the optimization problem. Furthermore, we also show that the mathematical models with and without heat loss at the end of the rod can both describe the experimental data well, and the outputs of the parameter estimation methods using these two models yield similar results.

Parameter estimation on the steady-state system using data gathered in the Instructional and Research Laboratory at NC State University (http://www.ncsu.edu/crsc/ilfum.htm) has been performed by Laguitao (2001). In our paper, aside from implementing our own experiments, we extended the steady-state model to the time-dependent case, implemented a numerical algorithm to solve the resulting PDE and tried two fundamentally different optimization algorithms for solving the parameter estimation problem.

MATERIALS AND METHODS

Heat Propagation Experiment
The experiment consists of two metallic rods the lengths of which are much greater than their radii. Dimensions of the two rods are shown in Table 1. A soldering iron in direct contact with one end of the rod was used to supply the heat source. We denote this end as the origin and we orient the $x$-axis along the length of the rod. Small holes sufficient to fit type-T thermocouples were drilled along the length of the rods. The locations of the thermocouples, denoted by $x_1$, $x_2$, ..., $x_7$, are given in Table 1. To record the thermocouple readings, we used a DI-1000TC data acquisition instrument (Dataq Instruments, Inc.) with eight channels measuring at most five samples per second on each channel. One channel was used to measure ambient temperature, thus only seven channels were left for the thermocouples. In Figure 1, we show the experimental setup in the Mathematical Modeling Laboratory of the Department of Mathematics, University of the Philippines Diliman.

Numerical computations were performed using Scilab, an open source software available at http://www.scilab.org/. The built-in algorithm leastsq was used for gradient-based optimization while we coded our own implementation of the genetic algorithm. We also implemented a Galerkin method for approximating the partial differential equation is presented in the results.

Mathematical Model
The mathematical analysis of the model is important to establish the validity of the computational results, but is not necessary to understand the results of this paper. Readers who do not wish to read the theoretical (but important) portions can skip the theoretical sub-sections and jump to the sub-sections where the parameter estimation problems for the steady-state and time-dependent cases are introduced.

We consider a one-dimensional heat flow (i.e., heat propagation along the radial direction is ignored), which is appropriate to the experimental setup since the length of the rods are much longer than the radii. The model of a one-dimensional heat flow along a cylindrical rod with radius $r$, length $\ell$, density $\rho$, specific heat capacity $c_p$, thermal conductivity $k$, and heat transfer coefficient $h$ is given by

$$c_p \frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{2h}{r} \left( u(x,t) - u_{\text{amb}} \right) \quad (1)$$

Table 1. Dimensions of the metallic rods and coordinates of the thermocouples.

<table>
<thead>
<tr>
<th>Rod Dimensions</th>
<th>copper</th>
<th>aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
<td>0.9980 (m)</td>
<td>1.003 (m)</td>
</tr>
<tr>
<td>radius</td>
<td>0.6375 (cm)</td>
<td>0.6250 (cm)</td>
</tr>
<tr>
<td>mass</td>
<td>1.1400 (kg)</td>
<td>0.3500 (kg)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coordinates of the Thermocouples (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>copper</td>
</tr>
<tr>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_3$</td>
</tr>
<tr>
<td>$x_4$</td>
</tr>
<tr>
<td>$x_5$</td>
</tr>
<tr>
<td>$x_6$</td>
</tr>
<tr>
<td>$x_7$</td>
</tr>
</tbody>
</table>

Figure 1. The experimental setup consisting of the metal rod, the heat source at the left end, the thermocouples and the data acquisition instrument.
where \( u(x,t) \) is the temperature of the rod at time \( t \) and location \( x \). The second term on the right-hand side of (1) is due to the heat loss at the side of the rod, and we have denoted by \( u_{amb} \) the ambient temperature. The heat source at \( x = 0 \) and heat loss at \( x = \ell \) yield the boundary conditions

\[
\begin{align*}
\frac{\partial u(0,t)}{\partial x} &= -\frac{Q}{k} \\
\frac{\partial u(\ell,t)}{\partial x} &= -\frac{h}{k} \left( u(\ell,t) - u_{amb} \right).
\end{align*}
\]

We denote the initial condition by

\[
u(x,0) = u_0(x).
\]

The list of symbols and units are given in Table 2. Our incorporation of the heat loss at the sides and at the end \( x = \ell \) follow from Newton’s law of cooling. The derivation of the heat equation, which is similar to the diffusion equation, may be found in many mathematical modeling or engineering textbooks (see for example, Widder (1975)) but we refer the readers to a detailed derivation of the modified heat equation (1) in Bargo (2007).

### Table 2. List of symbols and their units.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>temperature</td>
<td>K</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
<td>s</td>
</tr>
<tr>
<td>( x )</td>
<td>longitudinal direction</td>
<td>m</td>
</tr>
<tr>
<td>( r )</td>
<td>radius of the rod</td>
<td>m</td>
</tr>
<tr>
<td>( \ell )</td>
<td>length of the rod</td>
<td>m</td>
</tr>
<tr>
<td>( u_{amb} )</td>
<td>ambient temperature</td>
<td>K</td>
</tr>
<tr>
<td>( Q )</td>
<td>heat flux</td>
<td>W m(^{-2})</td>
</tr>
<tr>
<td>( \rho )</td>
<td>density of rod</td>
<td>kg m(^{-3})</td>
</tr>
<tr>
<td>( c_p )</td>
<td>specific heat of rod</td>
<td>J kg(^{-1})K(^{-1})</td>
</tr>
<tr>
<td>( k )</td>
<td>thermal conductivity of rod</td>
<td>W m(^{-1})K(^{-1})</td>
</tr>
<tr>
<td>( h )</td>
<td>heat transfer coefficient</td>
<td>W m(^{-1})K(^{-1})</td>
</tr>
</tbody>
</table>

### Well-posedness of the Model

To show well-posedness of a problem, we need to show existence, uniqueness, and continuous dependence of the solution to the initial value and parameter values. To accomplish this, we will pose the problem as an abstract Cauchy problem in an infinite-dimensional space and then use the Faedo-Galerkin method.

Let \( \Omega \) be the open subset \( (0, \ell) \subset \mathbb{R} \) with boundary \( \Gamma = \{0, \ell\} \). By multiplying both sides of Equation (1) by a test function \( \varphi \in H^1(\Omega) \), by integrating over the domain and using integration by parts, we obtain

\[
\begin{align*}
\rho c_p \int_0^\ell \frac{\partial u(x,t)}{\partial t} \varphi(x) dx &= -k \int_0^\ell \frac{\partial u(x,t)}{\partial x} \varphi'(x) dx + k u'(0,t) \varphi(0) - k u'(\ell,t) \varphi(\ell) \\
&\quad - \frac{h}{r} \int_0^\ell u(x,t) \varphi(x) dx + \frac{2h}{r} u_{amb} \int_0^\ell \varphi(x) dx.
\end{align*}
\]

For brevity, we denoted \( \frac{\partial u}{\partial x} \) by \( u' \). Imposing the boundary conditions in (2), we get

\[
\begin{align*}
\rho c_p \int_0^\ell \frac{\partial u(x,t)}{\partial t} \varphi(x) dx &= -k \left[ u'(0,t) \varphi(0) - u'(\ell,t) \varphi(\ell) \right] \\
&\quad + \frac{2h}{r} \int_0^\ell u(x,t) \varphi(x) dx + \frac{Q(t)}{r} \int_0^\ell \varphi(x) dx.
\end{align*}
\]

Now, let \( V = H^1(\Omega) \) and \( H = L^2(\Omega) \) with the following corresponding inner products:

\[
\begin{align*}
\langle \psi, \varphi \rangle_H &= \int_\Omega \psi(x) \varphi(x) dx \\
\langle \psi, \varphi \rangle_V &= \int_\Omega \psi'(x) \varphi(x) dx + \frac{2h}{r} \int_\Omega \psi(x) \varphi(x) dx.
\end{align*}
\]

It can be readily shown that our re-definition (6) of the inner product in \( V \) is equivalent to the usual Sobolev inner product in \( H^1(\Omega) \). Note also that \( V \) is continuously and densely embedded in \( H \), i.e., \( V \subset H \), and, thus, by identifying \( H \) with \( H^1(\Omega) \) through Riesz representation, we have the injections \( V \subset H \equiv H' \subset V' \). For \( \xi \in V' \) and \( \varphi \in V \), we denote the scalar product in the duality between \( V \) and \( V' \) by \( \langle \xi, \varphi \rangle_{V'} \). Then for any \( \xi \in H \), we have

\[
\langle \xi, \varphi \rangle_{V'} = \langle \xi, \varphi \rangle_H.
\]

The time derivative on the left hand side of (5) necessitates the notion of Hilbert space-valued derivatives in the distributional sense. We introduce the space \( W(0,T) \) as follows:

\[
W(0,T) = \left\{ f; f \in L^2(0,T; V), \frac{df}{dt} \in L^2(0,T; V') \right\}.
\]

It can be shown (see Lions 1971) that all functions \( f \in W(0,T) \) are, upon modification on a set of measure zero, continuous from \( [0, T] \) to \( H \), i.e.,

\[
W(0,T) \subset C^0([0,T]; H).
\]

We define the bilinear form \( \sigma: V \times V \rightarrow \mathbb{R} \) and the operator \( F: V \rightarrow \mathbb{R} \) by

\[
\begin{align*}
\sigma(\varphi, \psi) &= k \int_0^\ell \psi'(x) \psi'(x) dx + \frac{2h}{r} \int_0^\ell \psi(x) \psi(x) dx + \frac{Q(t)}{r} \int_0^\ell \psi(x) dx \\
F(\varphi) &= \frac{2h}{r} \int_0^\ell \varphi(x) dx + \frac{u_{amb}}{r} \varphi(\ell) + \frac{Q(t)}{r} \varphi(0).
\end{align*}
\]

Therefore, the weak form of the model is to look for \( u \in W(0,T) \) such that

\[
\begin{align*}
\left\{ \frac{du}{dt}, \varphi \right\}_{V'} &= -\sigma(u, \varphi) + F(\varphi), \quad \forall \varphi \in V \\
u(0) &= u_0 \in V.
\end{align*}
\]
It can be easily shown that the bilinear form $\sigma$ is continuous and $V$-elliptic. This means that there exists a unique bijective operator $A: V \rightarrow V^*$ such that $\sigma(\psi, \phi) = A(\psi)(\phi)$, for all $\psi, \phi \in V$. This allows us to equivalently write the weak form (10) as
\[
\begin{align*}
\left\{\begin{array}{l}
\frac{du}{dt} + Au + F, \quad (\text{in } V^*) \\
u(0) = u_0 \in V,
\end{array}\right.
\end{align*}
\]
and thus we have transformed the weak form into the abstract Cauchy problem
\[
\begin{align*}
\left\{\begin{array}{l}
\frac{du}{dt} = -Au + F, \quad (\text{in } V^*) \\
u(0) = u_0 \in V.
\end{array}\right.
\end{align*}
\] (12)

The functions $F$ and $\sigma$ in (9) do not depend on time, but to accommodate more general cases where $F$ and $\sigma$ are time-dependent, e.g., if the source function $Q$ or the ambient temperature $u_{amb}$ change with time, we will use arguments that hold for the general inhomogeneous abstract Cauchy problem. We will cite a general result from Lions (1971) that states the conditions needed for the solution of (12) to exist.

**Theorem 1.** Let $F \in L^2(0, T; V^*)$, and suppose that the following conditions are satisfied:

1. For all $\psi, \phi \in V$, the function $t \rightarrow \sigma(t; \psi, \phi)$ is measurable on $(0, T)$ and
   \[
   |\sigma(t; \psi, \phi)| \leq c\|\psi\|_{V'}\|\phi\|_{V}.
   \]

2. There exists $\lambda \in \mathbb{R}$, $\alpha > 0$ such that for all $\psi \in V$, $t \in (0, T)$,
   \[
   \sigma(t; \psi, \psi) + \lambda\|\psi\|^2_{H} \geq \alpha\|\psi\|^2_{V}.
   \]

Then, the problem (12) admits a unique solution in $W(0, T)$. Furthermore, the solution depends continuously on the data, i.e. the bilinear map $(F; u_0) \mapsto u$ is continuous from $L^2(0, T; V^*) \times C^0$ to $W(0, T)$. Since $F \in V^*$ is time-independent, then we can also say that $F \in L^2(0, T; V^*)$. Moreover, since the bilinear form $\sigma$ is continuous and $V$-elliptic, the assumptions of Theorem 1 are satisfied, hence, we conclude that the problem is well-posed.

**Parameter Estimation Problem with Steady-State Data**

At steady-state, the model (1) reduces to the boundary value problem
\[
0 = ku''(x) - \frac{2h}{r}(u(x) - u_{amb})
\]
\[
u'(0) = -\frac{Q}{k}
\]
\[
u'(t) = -\frac{h}{k}(u(t) - u_{amb})
\]
with general solution
\[
u(x) = c_1e^{-\beta x} + c_2e^{\beta x} + u_{amb},
\]
where $\beta = \sqrt{2h/rk}$. Imposing the boundary conditions enables us to solve for the constants $c_1$ and $c_2$:
\[
c_1 = \frac{Q(\beta - \frac{1}{\beta})}{k\beta(\beta + \frac{1}{\beta}) - \beta + \frac{1}{\beta} + \frac{Q}{k\beta}}
\]
and
\[
c_2 = \frac{Q(\beta - \frac{1}{\beta})}{k\beta(\beta + \frac{1}{\beta}) - \beta + \frac{1}{\beta} + \frac{Q}{k\beta}}.
\]

Let the vector $q$ contain the unknown parameters: $q = [Q, h, k]^T$. Given the steady-state temperature data $\{\hat{u}(x_1), \hat{u}(x_2), \ldots, \hat{u}(x_N)\}$ at $N$ points $x_1$, $x_2$, ..., $x_N$, the parameter estimation problem is
\[
\min_{q \in \Lambda} J(q) = \frac{1}{N} \sum_{i=1}^{N} \|u(x_i; q) - \hat{u}(x_i)\|^2
\]
where $u(x_i; q)$ is the solution of (13) using the parameter $q$ and evaluated at $x_i$, and $\Lambda$ is the set of admissible parameter values.

The steady-state temperature parameter estimation problem is useful for applications where temperature changes during the transient part are not so important. Note that the parameter $c_p$ could not be estimated since the term involving it vanishes in the steady-state problem. Furthermore, a closer inspection of the steady-state solution indicates that we cannot estimate all three parameters $Q, h$, and $k$, but rather only the ratios $Q/k$ and $h/k$. This is not an issue since the goal in model-based parameter estimation problems is to look for the effective values of the parameters wherein the model can function under different scenarios.

In our formulation of the parameter estimation problem (16), the data can contain measurement errors of the form
\[
\hat{u}(x_i) = u(x_i) + \varepsilon_i,
\]
where $u(x_i)$ is the actual temperature and $\varepsilon_i$, $\hat{u}(x_i)$ are independently and identically normally distributed errors.
We also would like to point out that due to the availability of an explicit solution to the steady-state problem, the computation needed to solve the parameter estimation problem is greatly reduced. Since initial guesses for parameter values are very crucial in performing the optimization problem, the steady-state problem can be used to find initial parameter values (except for $c_p$) for the time-dependent parameter estimation problem.

Parameter Estimation Problem with Time-dependent Data
For the time-dependent case, we denote the data (which may contain measurement errors) by the set
\[
\{\hat{u}(x, t_j)\}_{j=1, 1=1}^{N, N_i}. \quad \text{The parameter estimation problem, as in the steady-state case, is also posed as the least-squares problem}
\]

\[
\min_{q \in \Lambda} J(q) = \frac{1}{N \cdot N_i} \sum_{j=1}^{N} \sum_{i=1}^{N_i} \left| u(x, t_j; q) - \hat{u}(x, t_j) \right|^2, \tag{18}
\]

where $u(x, t_j; q)$ is the solution of the model equation using the parameter $q$ and evaluated at the point $x$, and at time $t_j$. The vector $q$ contains the unknown parameters, $q = [Q, h, c_p, k]^T$.

It can be seen that (18) is a far more computationally intensive optimization problem than (16) since each time the algorithm chooses a set of parameters, the partial differential equation (1) must be numerically approximated. Thus, the choice of numerical algorithms for solving the optimization and partial differential equation is very important when dealing with the time-dependent problem.

Finite-dimensional Parameter Estimation Problem and Convergence
At this point, we have shown existence and uniqueness of the solution in an infinite-dimensional Hilbert space and we have set up the optimization problem arising from the parameter estimation problem. In this section, we briefly discuss the framework for analyzing the convergence of the (Galerkin type) numerical approximations to the infinite dimensional solution. Furthermore, the parameter estimation problems could also possibly involve infinite dimensional admissible parameter spaces and hence, we also need to discuss the convergence of the finite-dimensional parameter estimation problem.

Consider a sequence of finite-dimensional approximating subspaces $V^n \subset V$ of the state space $V$, and a sequence of finite-dimensional approximating subspaces $\Lambda^n \subset \Lambda$ of the parameter space $\Lambda$. The projection from $V$ to $V^n$ is denoted by $P^n$ (similarly $P^n_*$ is the projection of $V^*$ onto $V^n$). The finite-dimensional parameter estimation problem (with finite-dimensional state spaces and parameter spaces) is

\[
\min_{q \in \Lambda^n} J^n(q) = \frac{1}{N \cdot N_i} \sum_{j=1}^{N} \sum_{i=1}^{N_i} \left| u^n(x, t_j; q) - \hat{u}(x, t_j) \right|^2, \tag{19}
\]

where $\hat{u}(x, t_j)$ is the observation at position $x$ and time $t_j$, and $u^n(q)$ is the solution to the finite-dimensional form of (10) given by

\[
\begin{cases}
\left\{ \frac{du^n}{dt}, \phi \right\}_{V^n, V^n} = -\sigma(u^n, \phi) + F(\phi), \forall \phi \in V^n \\
u^n(0) = P^n u_0.
\end{cases} \tag{20}
\]

Let us first consider the convergence of numerical approximations $u^n$ to the solution of (10). If the following properties are satisfied by the approximating spaces $V^n$:

(H1) $V^n \subset V$;

(H2) for each $\phi \in V, \|P^n_0 \phi\| \to 0$, as $n \to \infty$

(H3) the spaces satisfy the monotonicity condition $V^n \subset V^{n+1}$,

then the convergence of the finite-dimensional system to the infinite dimensional solution is guaranteed by the following theorem from Banks et al. (1996):

**Theorem 2.** Under the assumptions (H1)-(H3), the sequence $u^n$ converges to $u \in C[0, T; H]$, where $u^n$ is the solution of (20) and $u$ is the unique solution of (10).

Now, let us consider the convergence of the finite-dimensional parameter estimation problem. This has been discussed in Banks et al. (1996) for models of smart material structures where the following requirement for the finite-dimensional parameter spaces is specified:

(H4) The sets $\Lambda$ and $\Lambda^n$ lies in a metric space $\overline{\Lambda}$ with metric $d$, and are compact in this metric. Moreover, there is a mapping $i^n : \Lambda \to \Lambda^n$ such that $\Lambda^n = i^n(\Lambda)$, and $i^n$ converges to the identity on $\Lambda$.

We refer the readers to Theorem 5.1 in Banks et al. (1996), which guarantees the convergence of the sequence of parameter estimates $\{q_i\}$ to a solution of the finite-dimensional problem. Furthermore, Theorem 5.2 in that reference gives a general convergence result for parameter estimation problems.
RESULTS
In this section, we present our numerical approximations of the modeling equation and the estimated parameter values. The implementation of the numerical method for solving the partial differential equation is detailed in the first sub-section and we present our numerical results for the steady-state and time-dependent parameter estimation problems in the second and third sub-sections.

We used two different optimization algorithms to solve the parameter estimation problems: Scilab’s Newton-type (gradient-based) least squares optimization routine (leastsq command), and a genetic algorithm that we implemented in Scilab. These two algorithms can be easily found in textbooks, hence we do not discuss them in detail. Newton type algorithms have the advantage of quadratic convergence once the initial guess is close to the optimal point, but it is computationally expensive since the Jacobian of the system must be computed at each step of the iteration. Furthermore, it can be trapped in local minima and the possibility of non-convergence exists if an unsuitable initial guess is chosen. Genetic algorithms, on the other hand, are good at escaping local minima and though their convergence properties are not mathematically established, they have shown good performance in practice. A general review of quasi-Newton methods can be found in Dennis & Moore (1977), while a detailed description of the genetic algorithm can be found in Haupt & Haupt (2004).

A Galerkin Method for Numerical Approximation of the PDE
As mentioned earlier, the time-dependent model has no analytical solution and, hence, a numerical approximation algorithm is needed. We chose the Galerkin method, which is very similar to the finite-element method for one-dimensional problems. A crucial issue in the implementation of Galerkin methods is the choice of basis functions and since the weak form of the model (see Equation (5)) involves only one derivative, the use of linear splines is appropriate. To define the splines, we partition the interval \([0, \ell]\) with subintervals of length \(h_x = \ell / m\), and the \(m+1\) linear splines are given by

\[
\varphi_i(x) = \begin{cases} 
\frac{x - x_{i+1}}{h_x}, & x \in [x_{i+1}, x_i] \\
\frac{x_{i+1} - x}{h_x}, & x \in [x_i, x_{i+1}] \\
0 & \text{otherwise}
\end{cases}
\]  

(21)

The finite dimensional subspace \(V^n\) (where \(n = m + 1\)) is the space spanned by these linear basis functions. For each \(n\), the finite-dimensional problem is to look for \(u^n\) such that

\[
\left\{ u^n, \varphi^\alpha \right\}_H = -\alpha^n \left( u^n, \varphi^\alpha \right) + F^n(\varphi^n), \forall \varphi^n \in V^n
\]  

(22)

Writing the solution in \(V^n\) as

\[
u^n(x,t) = \sum_{i=1}^n \alpha_i(t) \varphi_i(x)
\]

(23)

and by substituting (23) in (22), we obtain a system of linear ordinary differential equations

\[
M \ddot{\alpha}(t) = A \alpha(t) + F(t)
\]

(24)

The mass matrix \(M\), stiffness matrix \(A\), and forcing vector \(F\) are given by

\[
\alpha(t) = [\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)]
\]

\[
A_i = -\frac{k}{\rho c_p} \int \frac{\varphi_i'(x)\varphi_j'(x)}{\rho c_p^2}dx - \frac{2h}{\rho c_p} \int \frac{\varphi_i(x)\varphi_j(x)}{\rho c_p}dx - \frac{h}{\rho c_p} \int \varphi_i(t)\varphi_j(t)
\]

\[
M_i = \int \varphi_i(x)\varphi_i(x)dx
\]

\[
F_i = \int \frac{2h u_n}{\rho c_p} \varphi_i(x)dx + \frac{\dot{Q}}{\rho c_p} \varphi_i(0).
\]

(25)

The initial condition for (24) is obtained by discretizing the initial condition \(u_0(x)\) in the following manner

\[
u^n(x,0) = \sum_{i=1}^n \alpha_i(0) \varphi_i(x) = u_0(x)
\]

(26)

We computed the values of the \(n\) unknowns \(\alpha_0 = [\alpha_1(0), \alpha_2(0), \ldots, \alpha_n(0)]\) in (26) using cubic spline interpolation on the initial temperature data. By interpolating at the grid points \(x_i\), the vector of initial conditions can be computed via

\[
\alpha_i(0) = u_0(x_i)
\]

(27)

The mass matrix \(M\) is invertible since it is symmetric and diagonally dominant (this usually arises from the choice of linear splines as basis function), hence we obtain the Cauchy problem

\[
\alpha(t) = \Lambda \alpha(t) + \tilde{F}(t)
\]

\[
\alpha(0) = \alpha_0,
\]

(28)

where \(\Lambda = M^{-1}A\) and \(\tilde{F}(t) = M^{-1}F(t)\). The system (28) is stiff, which we have verified by computing the difference of the magnitudes of the real parts of the eigenvalues of \(\Lambda\). In solving the system (28), we use the built-in stiff numerical ODE solver of Scilab.

Steady-state Parameter Estimation Results
To test whether the realistic boundary condition (2) has a significant effect on the model output, we also considered the non-realistic boundary condition at \(x = \ell\) without heat loss. This is given by

\[
\frac{\partial u(\ell,t)}{\partial x} = 0
\]

(29)
We will denote this non-realistic boundary condition as BC1, while we will denote condition (2) with heat loss at $x = \ell$ as BC2.

In Table 3, we present the optimal parameters computed by the least squares and genetic algorithms for both boundary conditions. Only the ratios (and not the actual values) of parameters $Q/k$ and $h/k$ could be estimated from the model. Our numerical results indicate that at steady-state, the two boundary conditions (BC1 and BC2) yield almost the same values of the parameters. Furthermore, the two numerical optimization algorithms converged to the same optimal values. The values of the objective function (16) reported in Table 3 show that the two optimization algorithms could both solve the optimization problem, and that the two models could equally fit the steady-state data.

In Figure 2, we plot the steady-state temperature of the two rods at each of the seven thermocouple locations. We also plot the steady-state model output using parameter values from the four cases (two boundary conditions and two optimization algorithms). The figure shows that the four numerical runs overlap and are in good agreement with the data.

**Table 3.** Parameters estimated using steady-state data and the steady-state model. Two boundary conditions were used: a non-realistic boundary condition (BC1) with no heat loss at the end $x = \ell$, and a realistic boundary condition (BC2) which incorporates heat loss at $x = \ell$. The parameter estimates from the two different optimization algorithms are presented.

<table>
<thead>
<tr>
<th></th>
<th>leastsq</th>
<th>GA</th>
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<tbody>
<tr>
<td></td>
<td>BC1</td>
<td>BC2</td>
</tr>
<tr>
<td>$Q/k$ (K/m)</td>
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<tr>
<td>copper</td>
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<td>aluminum</td>
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<td>$h/k$ (1/m)</td>
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<td>copper</td>
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<td>0.0354</td>
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<tr>
<td>aluminum</td>
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The time-series data and model output for the copper rod are plotted in Figure 3, while those for the aluminum rod are in Figure 4. In both plots, the model output using the two algorithms and two boundary conditions overlap (the estimated parameters for each of the four models are in Table 4) and are indistinguishable. This result agrees with the values of the objective functions in Table 4, and thus it seems that both the Newton-like optimization method and the genetic algorithm are appropriate for solving the optimization problem. Furthermore, it seems that heat loss at the end $x = \ell$ can be neglected in the model formulation.

The book values of $c_p$ and $k$ are available and for copper, the values are $c_p = 385 \text{ J/kg}^{-1}\text{K}^{-1}$ and $k = 370 \text{ W/m}^{-1}\text{K}^{-1}$. Thus, the book value ratio $385/370 = 1.041$ for copper is well estimated by the values in Table 4 of around 1.18. We also obtained good agreement between the book values and our estimates of the aluminum parameters. The book values are $c_p = 897 \text{ J/kg}^{-1}\text{K}^{-1}$ and $k = 237 \text{ W/m}^{-1}\text{K}^{-1}$ yielding a ratio of 3.785; this is close to our estimates that are around 3.9 (see Table 4).

**DISCUSSION**

We have shown that for the one-dimensional linear model of heat propagation on copper and aluminum rods, modeling the heat loss at the end of the rod away from the heat source produces the same output as a model without heat loss. We exhibited this by considering both steady-state and time-dependent models, and by using two different optimization algorithms to solve the nonlinear least-squares problem: a Newton-type method and the genetic algorithm method. For future work, this finding must be verified by employing a better heat source since our low-cost experiment used a soldering iron as a heat source, which possibly produced a non-constant flux. The use of a heat source with a thermostat will also reduce the number of unknowns in the model since the heat flux $Q$ will now be a known quantity.

Using the steady-state model, we were able to estimate two ratios of parameters, $Q/k$ and $h/k$, while in the time-dependent model, we were able to estimate an additional ratio $c_p/k$ and for this ratio, the obtained good agreement with the book values obtained from literature.
Figure 2. Steady-state data and model output using optimal parameter values: (top) copper rod and (bottom) aluminum rod. The points are steady-state temperature readings at the 7 thermocouples. The steady-state outputs of the 4 models are indistinguishable in the plot.
Although we were not able to explicitly obtain the values of each of the four parameters, our method is sufficient for applications requiring mathematical models of heat conduction since the model could predict the data well.

The experiment we presented may be easily modified to handle different boundary conditions. For instance, one could include in the model the fact that the flux provided by the soldering iron changes through time. This would involve a more complicated optimization problem since one of the unknowns is now a function. For real-life experiments wherein the heating of the rod is not in a controlled environment (such as in an air-conditioned room), we could also incorporate in the model the change in the ambient temperature through time.

The experiment could also be used to test the performance of other optimization algorithms such as simulated annealing, neural networks or hierarchical Bayesian methods. The design of a more efficient numerical method for solving the time-dependent model

Table 4. Parameters estimated using time-dependent data and the time-dependent model. Two boundary conditions were used: a non-realistic boundary condition (BC1) with no heat loss at the end $x = \ell$ and a realistic boundary condition (BC2) which incorporates heat loss at $x = \ell$. The parameter estimates from the two different optimization algorithms are presented.

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<tr>
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<tbody>
<tr>
<td></td>
<td>BC1</td>
<td>BC2</td>
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<td>Q/k (K/m)</td>
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<tr>
<td>aluminum</td>
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</table>

Figure 3. Model output for the copper rod using optimal parameters obtained from time-dependent data. The time-series for thermocouples located at $x_1$ to $x_7$, labeled TC1 to TC7, respectively, are plotted with the time-series data. Each model output (colored lines) is actually the plot of 4 models with different optimal parameters (Table 4), but the plots overlap and are indistinguishable indicating that the two boundary conditions and the parameters obtained from the two different algorithms can all describe the data well.
is also a research direction that can be explored using the experiment.

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(Available at the UP Diliman College of Science library) 18 p.


SCILAB: The open source platform for numerical computations. www.scilab.org
